THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224 – 5615
2007

A MODEL OF THE MECHANICAL WAVES IN CURVED BARS

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ABSTRACT

The different classical models of the mechanical waves in bars cover every a single type of the wave, respectively only longitudinal, transversal, inflexion or torsion wave and only in a rectilinear bar, every with his own hypothesis, physical and mathematical description and solving methods. Into a curved bar a certain wave changes his form along of the bar and transformed par example from a longitudinal wave in one or more types of the another wave, dependent of the form of the bar, ramie or connection with the environment. This work presents a physical and mathematical model of the mechanical wave of any type produced and conveyed in curved bars and his solving possibilities.

1. Introduction

A rectilinear bar excited periodical at a certain place (par example at his left end, figure 1) produces and conveys waves, dependent of the excitation and the bindings with the environment. Hereby appears longitudinal fig. 1a), transversal fig. 1b), inflexion fig. 1c) or torsion waves figure 1d). The mathematical model of these waves is known and ample described in [1].

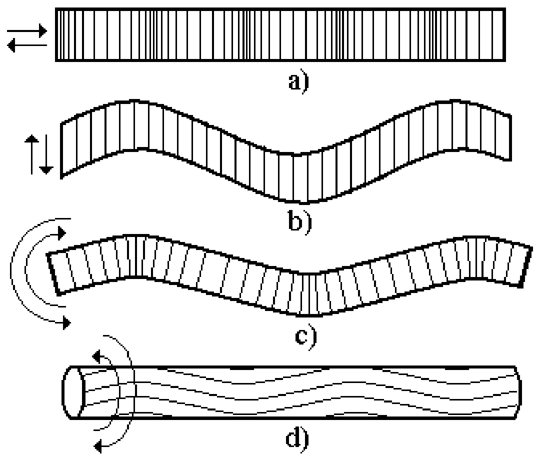

Figure 1. Waves in rectilinear bar

2. Wave in curved bars

If the bar is not linear, the wave maintains not more his original form from the excited point. Along of the curve appear geometrically modifications of the conduction medium which transform a longitudinal wave, par example, in more another waves through changes of the directions of the elastically deformations in ratio with the axis of the bar and through multiple reflections on the lateral surfaces, how these appear very nearly in fig. 2.

Figure 2. The modification of the form of the wave in a curved bar

Not a physically or mathematically model describes suchlike waves. In order to obtain a usable mathematical model we need a few hypotheses:

-the bar must have a finite transversal section, constant or variable along of the bar;

-the lateral surfaces of the bar have a known form and equation;

-the lateral surfaces of the bar have known connections with the environment.

The mathematical model of the wave can be written based on the generally differential equations Navier-Cauchy [2] in Cartesian coordinates, written in dynamical balance:

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + g_x = \rho \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + g_y = \rho \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + g_z = \rho \frac{\partial^2 w}{\partial t^2} \end{cases} \quad (1)$$

where $\sigma_x, \sigma_y, \sigma_z$ = normal stress, $\tau_{xy}, \tau_{yz}, \tau_{zx}$ = tangential stress with the properties $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$, g_x, g_y, g_z = mass forces, u, v, w = the absolute displacement on the coordinate rectangular axis, ρ = density, t= time. On the lateral surface of the bar appear the boundary conditions, respectively the influence of the environment:

$$\begin{cases} \sigma_x l + \tau_{xy} m + \tau_{xz} n = p_x \\ \tau_{yx} l + \sigma_y m + \tau_{yz} n = p_y \\ \tau_{zx} l + \tau_{zy} m + \sigma_z n = p_z \end{cases} \quad (2)$$

with *l, m, n* components on *x, y, z* axis of the unit vectors of the perpendiculars on the lateral surfaces, p_x, p_y, p_z the pressure of the environment on the lateral surfaces projected on the axis *x, y, z*.

The upper equations are completed with the equations dependent from the constitution of the material of the bar and express the dependence between the deformations and stress. For bars from steel these dependence are written [2]

$$\begin{cases} \sigma_x = 2G\left[\varepsilon_x + \frac{3\mu}{1-2\mu}\varepsilon_m\right] \\ \sigma_y = 2G\left[\varepsilon_y + \frac{3\mu}{1-2\mu}\varepsilon_m\right] \\ \sigma_z = 2G\left[\varepsilon_z + \frac{3\mu}{1-2\mu}\varepsilon_m\right] \\ \tau_{xy} = G\gamma_{xy}; \tau_{yz} = G\gamma_{yz}; \tau_{zx} = G\gamma_{zx} \end{cases} \quad (3)$$

where $\varepsilon_x, \varepsilon_y, \varepsilon_z$ are the longitudinal relative deformations, $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are the transversal relative deformations, G=shear modulus, μ=Poisson factor, ε_m the average deformation

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} \quad (4)$$

The shear modulus can be expressed

$$G = \frac{E}{2(1+\mu)} \quad (5)$$

with E = modulus of the elasticity. The relative longitudinal and transversal deformations can be written as functions from absolute displacements u, v, w

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x}; \quad \varepsilon_y = \frac{\partial u}{\partial y}; \quad \varepsilon_z = \frac{\partial u}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{cases} \quad (6)$$

Through a chain substitution in the equations from (1) to (6) we obtain a differential equations system from second order with unknown functions u, v, w, as functions dependent from the coordinates *x, y, z,* time *t*, mechanical and physical properties of the material, primary and boundary conditions, geometry of the bar. The mechanical and physical properties of the material change only the equations (3) and the value of the density ρ.

The hereby-obtained equations system is very difficult to solve on analytical way and therefore I have chosen a numerical way in order to find the solutions as four variable functions

$$u(x, y, z, t), v(x, y, z, t), w(x, y, z, t),$$

respectively the finite differences method.

The entire bar is divided in very little finite domains and the time is alike divided in very little intervals. The domains are numerated along of the each axis and have the index number ix, iy, iz, it on the axis *x, y, z* and time t, $ix = 1,2,\cdots,nx$, $iy = 1,2,\cdots,ny$, $iz = 1,2,\cdots,nz$, $it = 1,2,3.$ The numbers nx, ny, nz are the maximal values of the index number. The continuous functions $u(x, y, z, t)$, $v(x, y, z, t)$, $w(x, y, z, t)$, become discrete functions:

$$
\begin{aligned}
&u(x,y,z,t),\, v(x,y,z,t),\, w(x,y,z,t) \rightarrow \\
&\rightarrow u_{ix,iy,iz,it},\, v_{ix,iy,iz,it},\, w_{ix,iy,iz,it}
\end{aligned} \tag{7}
$$

respectively on the discontinuous domain ix, iy, iz and at the time interval it.

All derivatives from the differential equations system are in ratio with *x, y, z* or time *t*. Generally, the finite differences method transforms the derivatives in finite differences from more types, respectively backward, forward or central differences.

The derivatives in ratio with the time appear only as derivative from second order and I have chosen backward differences because always are known all anterior solutions. The differential ∂t becomes a little finite value written δt and the derivatives in ratio with the time *t* become

$$
\frac{\partial^2 u(x,y,z,t)}{\partial t^2} \rightarrow \frac{u_{ix,iy,iz,3} - 2u_{ix,iy,iz,2} + u_{ix,iy,iz,1}}{(\delta t)^2}
$$

$$
\frac{\partial^2 v(x,y,z,t)}{\partial t^2} \rightarrow \frac{v_{ix,iy,iz,3} - 2v_{ix,iy,iz,2} + v_{ix,iy,iz,1}}{(\delta t)^2}
$$

$$
\frac{\partial^2 w(x,y,z,t)}{\partial t^2} \rightarrow \frac{w_{ix,iy,iz,3} - 2w_{ix,iy,iz,2} + w_{ix,iy,iz}}{(\delta t)^2}
$$

The index number $it = 3$ corresponds to the current moment and index numbers $it = 2$, $it = 1$ correspond to the first and second previous moments. The derivatives in ratio with the coordinate's axis appear from first and second order. I have chosen only central and symmetrical differences in ratio with the coordinates and obligatory at the first previous moment, $it = 2$, in order to obtain only explicit equations in ratio with the currently displacements $u_{ix,iy,iz,3}$, $v_{ix,iy,iz,3}$, $w_{ix,iy,iz,3}$. The derivatives from first order becomes in finite differences:

$$
\begin{aligned}
&\frac{\partial u}{\partial x} \rightarrow \frac{u_{ix+1,iy,iz,2} - u_{ix-1,iy,iz,2}}{2(\delta x)} \\
&\frac{\partial u}{\partial y} \rightarrow \frac{u_{ix,iy+1,iz,2} - u_{ix,iy-1,iz,2}}{2(\delta y)} \\
&\frac{\partial u}{\partial z} \rightarrow \frac{u_{ix,iy,iz+1,2} - u_{ix,iy,iz-1,2}}{2(\delta z)}
\end{aligned} \tag{8}
$$

and the derivatives from second order becomes:

$$
\begin{cases}
\dfrac{\partial^2 u(x,y,z,t)}{\partial x^2} \rightarrow \\
\rightarrow \dfrac{u_{ix+1,iy,iz,2} - 2u_{ix,iy,iz,2} + u_{ix-1,iy,iz,2}}{(\delta x)^2} \\
\dfrac{\partial^2 v(x,y,z,t)}{\partial y^2} \rightarrow \\
\rightarrow \dfrac{v_{ix,iy+1,iz,2} - 2v_{ix,iy,iz,2} + v_{ix,iy-1,iz,2}}{(\delta y)^2} \\
\dfrac{\partial^2 w(x,y,z,t)}{\partial z^2} \rightarrow \\
\rightarrow \dfrac{w_{ix,iy,iz+1,2} - 2w_{ix,iy,iz,2} + w_{ix,iy,iz-1,2}}{(\delta z)^2}
\end{cases} \tag{9}
$$

The mixed derivatives calculated for example

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial x \;\partial y} \rightarrow \\
&\rightarrow \frac{\partial}{\partial y}\left[\frac{\left(u_{ix+1,iy,iz,2} - u_{ix-1,iy,iz,2}\right)}{2(\delta x)}\right] = \\
&= \frac{\left(u_{ix+1,iy+1,iz,2} - u_{ix-1,iy+1,iz,2}\right)}{4(\delta x)(\delta y)} - \\
&- \frac{\left(u_{ix+1,iy-1,iz,2} - u_{ix-1,iy-1,iz,2}\right)}{4(\delta x)(\delta y)} \rightarrow \\
&\rightarrow \frac{\partial^2 u}{\partial y \;\partial x}
\end{aligned} \tag{10}
$$

The other mixed derivatives are to calculate analogous to (10).

3. Example

I have chosen a plane curved bar (*z=0*), figure 3, excited at his left end.

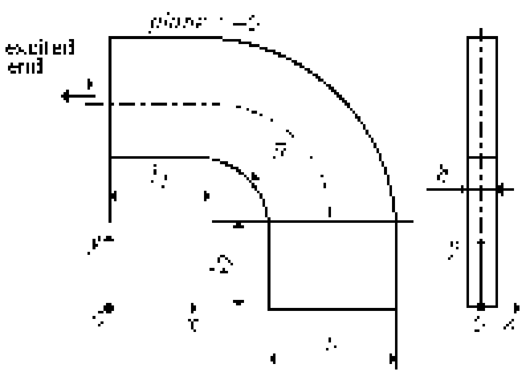

Figure 3. The curved bar

In order to maintain the Cartesian coordinates and the upper described equations on entire bar I have chosen the division of the bar only in rectangular finite elements, including the curved domain, figure 4.

The transversal section of the bar is

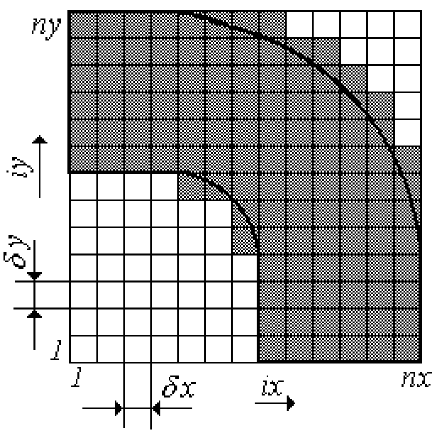

Figure 4. The finite elements

rectangular and contains a single row of finite elements $h = \delta z$ on the origin of the axis z, figure 2. In this case the axis Oz is not in use and the derivatives in ratio with this axis appear not more in the equations. The finite elements presented in figure 4 are many greater as in reality, in order to see clearly how is approximated the bar on the curved domain. In reality the approximation is better as in figure 4.

The dimensions of the bar are: $l_1 = 50$ mm, $l_2 = 50mm$, $R = 15mm$, $b = 10mm$, $h{=}1mm$, $\rho{=}7850Kg/m^3$, $E = 2.1 \cdot 10^{11}\ N/m^2$. The mass forces g_x, g_y, g_z are negligible.

The left end of the bar is harmonically excited with the function:

$$\begin{cases} u_{ix=1,\, iy,\, 3} = A \sin(\omega t) \\ v_{ix=1,\, iy,\, 3} = 0 \end{cases} \tag{11}$$

with $A{=}0.1\ mm$, $\omega{=}1250000s^{-1}$, respectively a high frequency chosen in order to produce in the bar more wavelength. The equation (11) represents withal the boundary conditions on the left side of the bar.

The finite elements of the bar have the value $\delta x = 1mm$, $\delta y = 1mm$, $\delta t = (2\pi)/(1000\omega)$ sec, namely thousand times littler as the oscillation period, in order to assurance the stability of the computation [4]. The calculation length is 2 periods of the excitation.

In computation process are counted only the dark designed finite elements, the limit values of the index numbers ix and iy at boundary are calculate adequately.

On boundary surfaces must be used the equations (2) but this fact is very difficult because of the numerous corners produced from the finite domains. I have chosen a easier way, respectively I consider the entire rectangular domain from $n_y \times n_x$ finite elements. The linear boundary surfaces are easy in ratio with equations (2) and the outsider elements from bar (white elements, figure 4) are very soft in ratio with the bar, $\rho{=}0.1\ Kg/m^3$, $E = 0.1 N/m^2$.

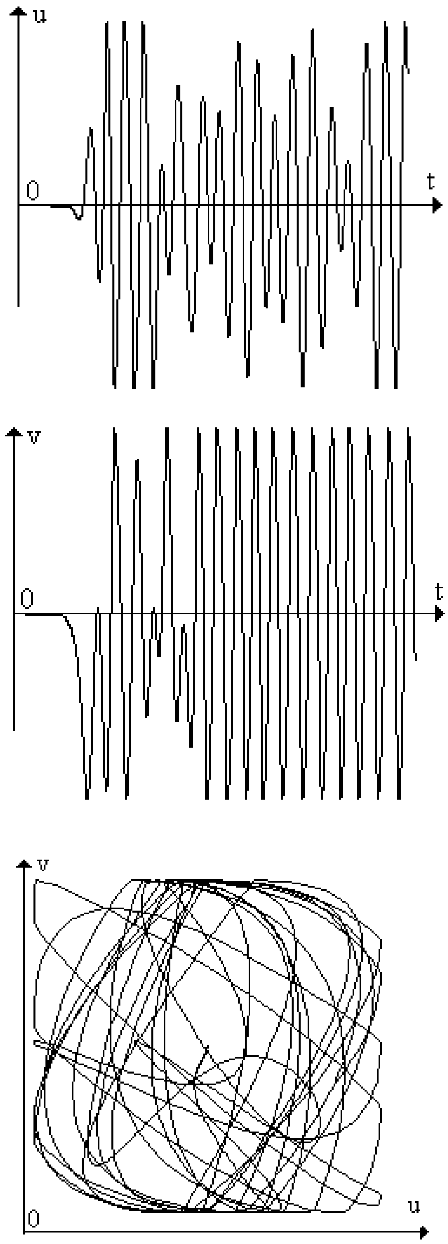

Figure 5. Oscillations and movement trajectory in the point $i_x = 10, i_y = 70$

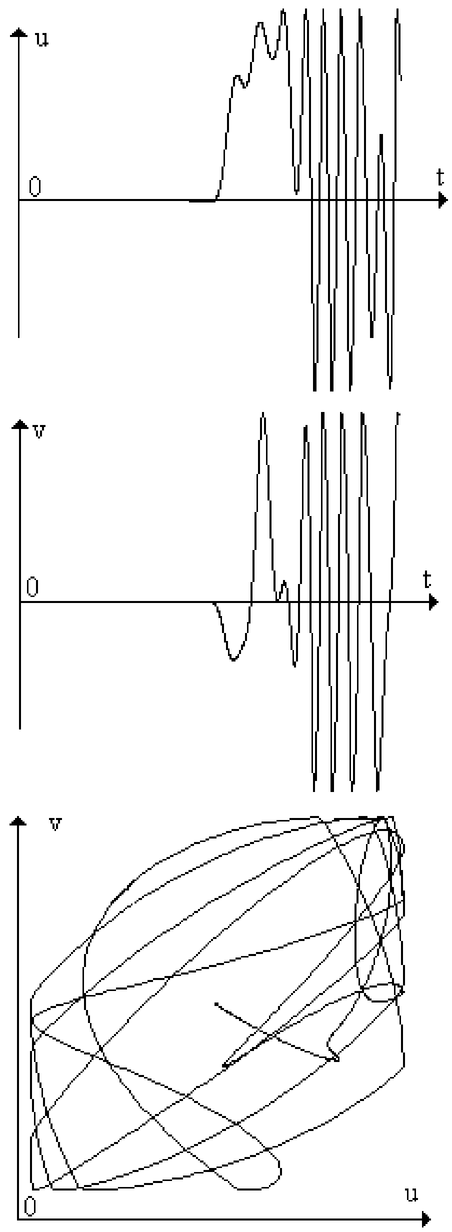

Figure 6. Oscillations and movement trajectory in the point $i_x = 70, i_y = 32$

In the figure 5 and 6 are presented the oscillations on the different points of the bar and the trajectory of the material particle from these points. Near excited end, figure 5, the movement begins early and the trajectory is influenced from direct waves and many reflections on more surfaces with more different angles, especially on the curved domain.

Near other end, figure 6, the movement begins later and the trajectory shows a dominant direction because reflections only on a few certain surfaces as majority cause. In another any point the movement and trajectory depend from the directions of the reflected and direct waves. The amplitude can be greater or lower, dependent from the phase ratio of the arrived waves. In the upper cases the maximal amplitude is 0.25 mm, constant on all directions.

4. Conclusions

The finite difference method solves the problems of the dynamic easier and faster as finite element method, because this last method uses very large equations system. I have applied the finite difference method under the form of the explicit equations, in order to obtain directly the currently solution in each point of the bar. I have encircled the bar with a rectangular domain, easy to be solved on the boundary surfaces and the addition material are properties which not influences the comportment of the bar. Disadvantageous is the hazard of the instability of the solution, resolved with very little time finite difference and the low precision, resolved with more finite elements.

The vibrations into curved bar appear under multiple influences, respectively direct wave and more reflected waves on any possible surfaces. The interference between these different waves can produces the increase or decrease of the resulted oscillation, different from the point to point.

The results are obtained on a Pentium IV PC programmed from the author in DELPHI 6.

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