DYNAMIC ANALYSIS OF MECHANICAL SYSTEMS. 
CALCULUS OF THE EQUIVALENT DYNAMIC FORCES AND TORQUES

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ABSTRACT

This article presents the physical and mathematical models of mechanical systems with elatic and/or rigid elements, subjected to dynamical loads: forces and torques. The aim of the study is to establish some formulas easy to use to equate these dynamical loads in any point of the system. It had taken into account both the ideal mechanical systems (with no energy losses) and the real systems according to their mechanical efficiency.

KEYWORDS: mechanical systems, dynamics, equivalent forces and torques

1. Introduction

Trough dynamical loads it is understood the sum of the forces and couples of forces which operate on the system. These loads can be produced by:
- the driving motor(s); in this case, the load produces motor mechanical work
- the working devices of the equipments, the mechanical work being negative (required work)
- resistance loads like friction forces and torques; in this case the work is also negative (work of resistance/friction)

The method to reduce mechanical system and their loads to less complex systems consists in finding some mechanical features equivalent to primary ones. Being equivalent means for mechanical systems to have the same energy, power, to do the same work, to have the same natural frequencies and eigenvalues, aso. This article uses in order to equate forces and torques, the principle of the equivalency between the real forces and torques work and the work of the reduced/equivalent ones.

2. The general problem of forces and torques equation

It is considered a mechanical system subjected to \( n \) concentrated forces \( F_i, i=1,n \) and \( m \) torques \( M_j, j=1,m \). Also it is considered that infinitesimal displacements of the points of application of the forces are \( d\vec{r}_i \) \( i=1,n \) and the infinitesimal rotations of the couples are \( d\vec{\phi}_j \), where \( j=1,m \).

The total infinitesimal mechanic work of all forces and torques is the sum of the infinitesimal mechanic work of each of them:

\[
dL = \sum_{i=1}^{n} F_i d\vec{r}_i + \sum_{j=1}^{m} M_j d\vec{\phi}_j \quad (1)
\]

Considering that the displacements of the points of application of the forces are parallel to the forces and the rotations are in a perpendicular plane to the couples, each scalar product of the relation (1) is equal with the product of the scalars, therefore the total infinitesimal mechanic work becomes:

\[
dL = \sum_{i=1}^{n} F_i d\vec{r}_i + \sum_{j=1}^{m} M_j d\phi_j \quad (2)
\]
The products of each sum of relation (2) are positive for the motor work and negative for the mechanical work of resistance.

If it is necessary to calculate an equivalent force for the entire system, this force has to do the same mechanical work, meaning

\[ \mathbf{F}_{eqv}d\mathbf{s} = dL, \]

where \( d\mathbf{s} \) is vectorial displacement of the point of application of these forces.

Considering that the displacement of the point of application is parallel to the force, the relation (2) of the mechanical work becomes:

\[ F_{eqv}ds = \sum_{i=1}^{n} F_i ds_i + \sum_{j=1}^{m} M_j d\phi_j \]

(4)

If the relation (4) is divided by the infinitesimal time of motion \( dt \), the work is now

\[ F_{eqv} \frac{ds}{dt} = \sum_{i=1}^{n} F_i \frac{ds_i}{dt} + \sum_{j=1}^{m} M_j \frac{d\phi_j}{dt}, \]

(5)

or

\[ F_{eqv} = \sum_{i=1}^{n} F_i \frac{v_i}{v} + \sum_{j=1}^{m} M_j \frac{\omega_j}{v}, \]

(6)

where \( v \) is the speed of the equivalent force (where it is making the equation of the entire system of loads)

\( v_i \) - the speed of the point of application of the force \( F_i \quad i=1,n \)

\( \omega_j \) - the angular speed of the torque \( M_j \quad j=1,m \)

Dividing the relation (6) by \( v \), the calculus formula for the equivalent force is:

\[ F_{eqv} = \sum_{i=1}^{n} F_i \frac{v_i}{v} + \sum_{j=1}^{m} \frac{M_j \omega_j}{v} \]

(7)

To calculate an equivalent torque for the entire system of loads, this torque has to do the same infinitesimal mechanical work

\[ M_{eqv}d\phi = dL, \]

(8)

if the infinitesimal rotation vector is parallel to the torque vector. In this case the work becomes:

\[ M_{eqv}d\phi = \sum_{i=1}^{n} F_i ds_i + \sum_{j=1}^{m} M_j d\phi_j \]

(10)

Dividing the relation (10) by \( dt \), the infinitesimal mechanical work can be written as follows

\[ M_{eqv} \cdot \frac{\omega}{dt} = \sum_{i=1}^{n} F_i \frac{v_i}{\omega} + \sum_{j=1}^{m} M_j \frac{\omega_j}{\omega}, \]

(11)

where \( \frac{d\phi}{dt} = \omega \) is the rotational speed of the equivalent torque \( M_{eqv} \).

By dividing the work (11) through \( \omega \), it obtains the calculus formula for equivalent torque like this:

\[ M_{eqv} = \sum_{i=1}^{n} F_i \frac{v_i}{\omega} + \sum_{j=1}^{m} \frac{M_j \omega_j}{\omega} \]

(12)

3. The equivalent torque of the mechanical transmissions with gears

It considers a simple mechanical transmission with one step gear like in fig. 1.

Fig. 1 The calculus model of the one step gearing mechanical transmission

1-driving shaft
2-driven shaft
\( M_M \)-the moment/torque of the motor
\( M_{res} \)-the resistance moment/torque
\( \omega_1, \omega_2 \)-rotational speed of the shafts

Fig. 2 The calculus diagram for the equivalent moment of resistance

For the gear transmission from fig. 1,
the ratio is:

\[ i = i_{12} = \frac{\omega_1}{\omega_2} \]  

(13)

Figure 2 shows the calculus diagram of the equivalent moment of resistance only (the calculus diagram and the way to calculate the equivalent inertia are acc. to [1]). The calculus diagram for the equivalent motor moment is shown in the fig. 3.

\[ \text{Fig. 3 The calculus diagram for the equivalent driving moment} \]

### 3.1. Ideal gear transmission

If there are no losses in the step gear, the equivalent torques can be calculated by the relation (12) as follows:

1. the equivalent moment of resistance (calculated to the shaft no. 1)

\[ M_{\text{res eqv}} = M_{\text{res}} \frac{\omega_2}{\omega_1} = \frac{M_{\text{res}}}{i} \]  

(14)

2. the equivalent motor moment (calculated to the shaft no. 2)

\[ M_{\text{Meqv}} = M_M \frac{\omega_1}{\omega_2} = M_M \cdot i \]  

(15)

Since the ratio \( i \) can take different values, we may draw some intermediate conclusions:

1. if \( i = 1 \) (gears for changing the sense of rotation only) the both equivalent moment and the equivalent moment of resistance stays unchanged

2. if \( i > 1 \) (reduced step gear) the motor moment is amplified by \( i \) and the moment of resistance is divided/decreased by \( i \)

3. if \( i < 1 \) (amplifier step gear) the motor moment is divided by \( i \) and the moment of resistance is amplified by \( i \).

### 3.2. Gear with mechanical losses

To establish the analytical formulas of the equivalent moments for a transmission with one step gear with mechanical losses. It considers the diagram from fig. 1, where the ratio is done by (13) and the mechanical efficiency is \( \eta \). The motor moment is in the sense of \( \omega_1 \) and the moment of resistance is against the sense of \( \omega_2 \).

For the diagram shown in fig. 2, the power balance can be written

\[ -M_{\text{res eqv}} \omega_1 = -M_{\text{res}} \omega_2 - P_{\text{loss}} \]  

(16)

where \( P_{\text{loss}} \) is the loss of power in the gear.

If the loss of power is written as

\[ P_{\text{loss}} = M_{\text{loss}} \omega_1 \]  

(17)

where \( M_{\text{loss}} \) is the loss of moment in the gear, relation (16) becomes:

\[ (M_{\text{res eqv}} - M_{\text{loss}}) \omega_1 = M_{\text{res}} \omega_2 \]  

(18)

or

\[ M_{\text{res eqv}} - M_{\text{loss}} = \frac{M_{\text{res}} \omega_2}{\omega_1} \]  

(19)

In order to underline the influence of the mechanical efficiency of the gear, it divides (19) by \( \omega_1 \), obtaining:

\[ \frac{M_{\text{res eqv}} - M_{\text{loss}}}{M_{\text{res eqv}}} = \frac{M_{\text{res}} \omega_2}{\omega_1} \]  

(20)

Observing that in the left side of the equal sign is the mechanical efficiency \( \eta \), the relation (20) becomes

\[ \eta = \frac{M_{\text{res}} \omega_2}{M_{\text{res eqv}}} \]  

(21)

from which it is obtained the expression of the equivalent moment of resistance on the driving shaft 1:

\[ M_{\text{res eqv}} = \frac{M_{\text{res}} \omega_2}{\eta \cdot \omega_1} \]  

(22)

For equivalent motor moment calculation it considers the diagram from fig. 3. In this case, the power balance is as follows:

\[ M_{\text{Meqv}} \omega_2 = M_M \omega_1 - P_{\text{loss}} \]  

(23)

Taking in consideration the relation (17), the power balance can be written

\[ M_{\text{Meqv}} \omega_2 = (M_M - M_{\text{loss}}) \omega_1 \]  

(24)

or like this:
\[ M_{Meqv} \omega_2 = \frac{M_M - M_{\text{loss}}}{M_M} M_M \omega_1 \]  
(25)

Since the fraction from the right side of the equation (25) is the mechanical efficiency \( \eta \), the power balance can be written

\[ M_{Meqv} \omega_2 = \eta M_M \omega_1, \]  
(26)

from which it is obtained the equivalent motor moment on the driven shaft 2:

\[ M_{Meqv} = \eta M_M \frac{\omega_1}{\omega_2} = M_M \eta \]  
(27)

As the mechanical efficiency \( \eta < 1 \), it may say that, taking into consideration the mechanical losses from the gear leads to:
1\textsuperscript{a} a decreasing of the equivalent motor moment
2\textsuperscript{a} an increasing of the equivalent moment of resistance

### 4. The equivalent torques calculus for the driving mechanisms with gears

To understand the equation method for the torques, it is considered a simple electric mechanical driving of a working device like in fig. 4. It is assumed that the acting torques are:
- the driving/motor moment \( M_M \)
- the moment of resistance \( M_{Ir}, i = 1,4 \)
- the moment of the working device \( M_{WD} \)

![Fig. 4 The skeleton diagram of a driving mechanism with four steps gear](image)

EM - electromotor
WD - working device
1,2,3,4,5,6,7,8 - gear wheels (gw)
I - driving shaft
V - driven shaft
II,III,IV - intermediate motion shaft

Considering that the transmission shafts are rigid, the ratio of each step gear is:
- gear gw1-gw2

\[ \frac{\omega_{gw1}}{\omega_{gw2}} = \frac{\omega_1}{\omega_2} \]  
(28)

- gear gw3-gw4
\[ \frac{\omega_{gw3}}{\omega_{gw4}} = \frac{\omega_2}{\omega_3} \]  
(29)

- gear gw5-gw6
\[ \frac{\omega_{gw5}}{\omega_{gw6}} = \frac{\omega_3}{\omega_4} \]  
(30)

- gear gw7-gw8
\[ \frac{\omega_{gw7}}{\omega_{gw8}} = \frac{\omega_4}{\omega_5} \]  
(31)

### 4.1. The equivalent torques calculus on the motor shaft

If it’s necessary an operation of equation of all torques on the driven shaft \( I \), the calculus diagram is shown in fig. 5. In this case, it has to equate all the torques excepting the motor moment \( M_M \).

Neglecting the mechanical losses in the gear, the power balance is as follows:

\[ -M_{Meq} \omega_I = -M_{1\text{res}} \omega_I - M_{2\text{res}} \omega_2 - M_{3\text{res}} \omega_3 - M_{4\text{res}} \omega_4 - M_{WD} \omega_5 \]  
(32)

where all minus signs are done by the differences between the senses of the torques and the senses of the rotation speed.

If the relation (32) is multiplied by “-1” and divided by \( \omega_I \), it is obtained the sum of the equivalent torques on shaft 1 like this:

![Fig. 5 The calculus diagram for the equivalent moment on the driving shaft I](image)
Taking into consideration the definition of the gear ration done by the relations (28)-(31), the total equivalent moment on the motor shaft \( I \) can be written as follows:

\[
M_{1\text{eqv}} = M_{1\text{res}} + \frac{M_{2\text{res}}}{i_1} + \frac{M_{3\text{res}}}{i_1^2} + \frac{M_{4\text{res}}}{i_1^3} + \frac{M_{WD}}{i_1} \tag{33}
\]

In order to determine the calculus formula of the equivalent torques on the motor shaft taking into consideration the mechanical efficiency of the gears \( \eta_1, \eta_2, \eta_3, \eta_4 \), it uses the relation (22) because all the equivalent torques are resistant. In this case, the total equivalent torque on the motor shaft is:

\[
M_{1\text{eqv}} = M_{1\text{res}} + \frac{M_{2\text{res}}}{i_1} + \frac{M_{3\text{res}}}{i_1^2} + \frac{M_{4\text{res}}}{i_1^3} + \frac{M_{WD}}{i_1} \tag{34}
\]

In order to determine the calculus formula of the equivalent torques on the motor shaft taking into consideration the mechanical efficiency of the gears \( \eta_1, \eta_2, \eta_3, \eta_4 \), it uses the relation (22) because all the equivalent torques are resistant. In this case, the total equivalent torque on the motor shaft is:

\[
M_{1\text{eqv}} = M_{1\text{res}} + \frac{M_{2\text{res}}}{i_1} + \frac{M_{3\text{res}}}{i_1^2} + \frac{M_{4\text{res}}}{i_1^3} + \frac{M_{WD}}{i_1} \tag{35}
\]

### 4.2. The equivalent torques calculus on the working device shaft

Depending of the type of dynamical analysis of the mechanical transmission with gear, it can equate all the moments on the working device shaft. In this case, the simplified diagram is shown in fig. 6.

Neglecting the mechanical losses from gears, the power balance can be written like this:

\[
M_{5\text{eqv}} = -M_{4\text{res}} \omega_4 - M_{3\text{res}} \omega_3 - M_{2\text{res}} \omega_2 - M_{1\text{res}} \omega_1 - M_{M} \omega_1 \tag{36}
\]

If the relation (36) is divided by \( \omega_5 \), it is obtained the calculus formula for the total equivalent moment on the working device shaft \( V \) as follows:

\[
M_{5\text{eqv}} = -M_{4\text{res}} \frac{\omega_4}{\omega_5} - M_{3\text{res}} \frac{\omega_3}{\omega_5} - M_{2\text{res}} \frac{\omega_2}{\omega_5} - M_{1\text{res}} \frac{\omega_1}{\omega_5} - M_{M} \frac{\omega_1}{\omega_5} \tag{37}
\]

4.3. The equivalent torques calculus on intermediate motion shafts

If the equation of the moments has to be done on another shaft (than motor shaft or working device shaft), the process of equation will be done taking into consideration the angular speed of that shaft and the branch of the skeleton of the transmission.

For instance, if the torque equation is done on the intermediate shaft \( III \), the calculus diagram is that from fig. 7. The equivalent torques from that diagram are composed from:

\[
M_{5\text{eqv}} = \left( M_{M} - M_{1\text{res}} \right) \frac{i_1 i_2 i_3 i_4}{\eta_1 \eta_2 \eta_3 \eta_4} - M_{2\text{res}} i_2 i_3 i_4 - M_{3\text{res}} i_3 i_4 - M_{4\text{res}} i_4 \tag{38}
\]

Now, if we consider the losses of power in the gears by the mechanical efficiency of the gears, the calculus formula of the total equivalent moment on the working device gear \( V \) becomes:

\[
M_{5\text{eqv}} = \left( M_{M} - M_{1\text{res}} \right) \frac{i_1 i_2 i_3 i_4}{\eta_1 \eta_2 \eta_3 \eta_4} - M_{2\text{res}} i_2 i_3 i_4 - M_{3\text{res}} i_3 i_4 - M_{4\text{res}} i_4 \tag{39}
\]
$M^\text{reg}_{\text{seqv}}$ - the sum of the equivalent moments of
the right branch of the shaft \textbf{III} (working
device moment, resistant moment from the shaft
\textbf{IV})

Considering the nature of the moments
(motor, resistant), the power balance of the left
branch on the shaft \textbf{III} is:

$$M^\text{left}_{\text{seqv}} \omega_3 = -M_{3\text{res}} \omega_3 - M_{2\text{res}} \omega_2 -$$
$$- M_{1\text{res}} \omega_1 + M_M \omega_I$$

(40)

The calculus formula of the total
equivalent moment on left branch is obtained by
dividing the relation (40) through $\omega_3$:

$$M^\text{left}_{\text{seqv}} = -M_{3\text{res}} - M_{2\text{res}} \frac{\omega_2}{\omega_3} -$$
$$- M_{1\text{res}} \frac{\omega_I}{\omega_3} + M_M \frac{\omega_I}{\omega_3}$$

(40)

The relation giving $M^\text{left}_{\text{seqv}}$ can be
written taking into consideration the gears ratio
as follows:

$$M^\text{left}_{\text{seqv}} = (M_M - M_{1\text{res}}) i_2 -$$
$$- M_{2\text{res}} i_2 - M_{3\text{res}}$$

(40)

The power balance for the right branch
of the transmission is

$$- M^\text{reg}_{\text{seqv}} \omega_3 = -M_{4\text{res}} \omega_4 - M_{WD} \omega_5 .$$

(41)

or, by dividing trough $- \omega_3$:

$$M^\text{reg}_{\text{seqv}} = M_{4\text{res}} \frac{\omega_4}{\omega_3} + M_{WD} \frac{\omega_5}{\omega_3}$$

(42)

Relation (42) can be written function of
gear ratio as follows:

$$M^\text{reg}_{\text{seqv}} = \frac{M_{4\text{res}}}{i_3} + \frac{M_{5\text{res}}}{i_3i_4}$$

(43)

For the equation of the moments taking
into consideration the losses of power by the
mechanical efficiency of the gear ($\eta_1, \eta_2, \eta_3,
\eta_4$), it has to use relations (22) and (27). In
this case, the equivalent total moments on the
shaft \textbf{III} are:

- on left branch

$$M^\text{left}_{\text{seqv}} = M_M i_2 \eta_2 \eta_{12} - M_{1\text{res}} \frac{i_{12}}{\eta_2} -$$
$$- M_{2\text{res}} \frac{i_2}{\eta_2} - M_{3\text{res}}$$

(44)

- on right branch

$$M^\text{reg}_{\text{seqv}} = \frac{M_{4\text{res}}}{i_3 \eta_3} + \frac{M_{WD}}{i_3i_4 \eta_3 \eta_4}$$

(45)

5. Conclusions

This article describes the general
method of equation of statical or dynamic loads
applied to mechanical systems. Particularly,
there are the proceeding and the formulae for
the equivalent torques’ calculus of the
mechanical transmission with gears.

These methods and calculus formulae
are useful both to the design engineers and to
the dynamics experts. Also, the methods to
equate the torques can be useful to the students,
candidates for master’s and doctor’s degree.

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