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THE APPROXIMATE ANALYTICAL SOLUTIONS FOR THE DIFFERENTIAL EQUATIONS WHICH DESCRIBE FREE VIBRATIONS OF NON-LINEAR MECHANICAL SYSTEMS

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ABSTRACT

The oscillation movement of a mechanical non-linear system is not easy to solve exactly on analytical way. The approximate solutions are based on different methods and give different values with different approximation degree. Our method, based on "low parameter method" gives an another way for this almost known method and it is applicable at non-linear mechanical systems on free movement.

1. Introduction

Free oscillations of the non-linear mechanical systems are described by differential equations such as

$$m\ddot{x} + F_i = 0$$

where m is the mass of the system and F_i are the internal forces (elastic and damping).

The equation can be written

$$\ddot{x} + \mu\ f(x,\dot{x}) + x = 0 \qquad (1)$$

where μ is a low positive parameter, the f function includes linear and non-linear terms depending on $x, \dot{x}$ and coefficients depending of the mass of the system, x being the movement leaning on time.

2. The principle

Marking $\dot{x} = z, z = z(t)$ we obtain a differential equivalent system of the equation (1)

$$\begin{cases} \dot{x} = z \\ \dot{z} = -x - \mu\ f(x,z) \end{cases} \qquad (2)$$

The system (2) becomes under the disregarding of the non-liniarities ($\mu = 0$)

$$\begin{cases} \dot{x} = z \\ \dot{z} = -x \end{cases}$$

with the solutions

$$\begin{cases} x = C_1 \cos(t) + C_2 \sin(t) \\ z = -C_1 \sin(t) + C_2 \cos(t) \end{cases} \qquad C_1, C_2 \in R$$

For the non-linear system (2), we look for solutions such as

$$\begin{cases} x = C_1(t) \cos(t) + C_2(t) \sin(t) \\ z = -C_1(t) \sin(t) + C_2(t) \cos(t) \end{cases} \qquad (3)$$

the functions $C_1(t)$, $C_2(t)$ will be determined by identifying from the condition that (3) should be a solution for system (2). It results

$$\begin{cases} \dot{C}_1(t) \cos(t) + \dot{C}_2(t) \sin(t) = 0 \\ -\dot{C}_1(t) \sin(t) + \dot{C}_2(t) \cos(t) = \\ = -\mu\ f\begin{bmatrix} C_1(t) \cos(t) + C_2(t) \sin(t), \\ -C_1(t) \sin(t) + C_2(t) \cos(t) \end{bmatrix} \end{cases} \qquad (4)$$

By solving of the system (4) with the unknown $C_1(t)$, $C_2(t)$ it results

$$\begin{cases} \dot{C}_1(t) = \\ = \mu\ f\begin{bmatrix} C_1(t)\cos(t) + C_2(t)\sin(t), \\ -C_1(t)\sin(t) + C_2(t)\cos(t) \end{bmatrix} \\ \dot{C}_2(t) = \\ = -\mu\ f\begin{bmatrix} C_1(t)\cos(t) + C_2(t)\sin(t), \\ -C_1(t)\sin(t) + C_2(t)\cos(t) \end{bmatrix} \end{cases} \tag{5}$$

Integrating, we obtain $C_1(t), C_2(t)$ and coming back to (3) we find the exact analytical solution. But system (4) can be solved only in particular situations for simple expressions of the f functions.

For low non-linearities we consider that in a period 2π , the functions $C_1(t), C_2(t)$ remain invariable and can be approximate with their medium values on this period, and than

$$\dot{C}_1(t) = \frac{\mu}{2\pi}\int_0^{2\pi} f\begin{pmatrix} C_1\cos\sigma + C_2\sin\sigma\ , \\ -C_1\sin\sigma + C_2\cos\sigma \end{pmatrix}\sin\sigma\ d\sigma$$

$$\dot{C}_2(t) = -\frac{\mu}{2\pi}\int_0^{2\pi} f\begin{pmatrix} C_1\cos\sigma + C_2\sin\sigma\ , \\ -C_1\sin\sigma + C_2\cos\sigma \end{pmatrix}\cos\sigma\ d\sigma$$

After the calculating of the integral we obtain results as

$$\begin{cases} \dot{C}_1(t) = \mu\ g[C_1(t), C_2(t)] \\ \dot{C}_2(t) = \mu\ h[C_1(t), C_2(t)] \end{cases} \tag{6}$$

respectively a system of differential equations that can be integrated easier than (5) and its solutions lead to solving the non-linear equation.

3.Example

We will solve as a particular case the following non-linear equation

$$\ddot{x} + \mu\ \dot{x}^3 + x = 0 \tag{7}$$

so $f(\dot{x}, x) = \dot{x}^3$.

The expressions that help us obtain $C_1(t), C_2(t)$ can be written

$$\dot{C}_1(t) = \frac{\mu}{2\pi}\int_0^{2\pi}(-C_1\sin\sigma + C_2\cos\sigma)^3\sin\sigma\ d\sigma$$

$$\dot{C}_2(t) = -\frac{\mu}{2\pi}\int_0^{2\pi}(-C_1\sin\sigma + C_2\cos\sigma)^3\cos\sigma\ d\sigma$$

where $C_1, C_2 \in R$

After integrating we obtain

$$\begin{cases} \dot{C}_1(t) = -\dfrac{3\mu\ C_1}{8}\left(C_1^2 + C_2^2\right) \\ \dot{C}_2(t) = -\dfrac{3\mu\ C_2}{8}\left(C_1^2 + C_2^2\right) \end{cases} \tag{8}$$

The presence of $C_1^2 + C_2^2$ in the right part of the two equations leads to a division and it results

$$\frac{\dot{C}_1(t)}{\dot{C}_2(t)} = \frac{C_1}{C_2} \tag{9}$$

differential equation that conducts to a linear relation between functions $C_1(t), C_2(t)$

$$C_2(t) = kC(t), k \in R \tag{10}$$

It results

$$x(t) = C_1(t)\cos t + kC_1(t)\sin t = C_1(t)[\cos t + k\sin t]$$

the function $C_1(t)$ verifying

$$\begin{aligned} &\dot{C}_1(t) = -\frac{3\mu\ C_1(t)}{8}\left[C_1^2(t) + k^2 C_1^2(t)\right]; \\ &\dot{C}_1(t) = -\frac{3\mu}{8}\left(k^2 + 1\right)C_1^3(t); \\ &\dot{C}_1(t) + \omega^2 C_1^3(t) = 0 \end{aligned} \tag{11}$$

Noting $z(t) = C_1^{-2}(t)$ it results later

$z(t) = 2\omega^2 t + c, c \in R$, $\omega = \dfrac{3\mu}{8}\left(k^2 + 1\right)$ and than

$$C_1(t) = \pm \frac{1}{\sqrt{2\omega^2 t + c}}$$

Choosing the solution with the sign + (with the other we proceed identically), we obtain

$$x(t) = \frac{1}{\sqrt{2\omega^2 t + c}}(\cos t + k \sin t) \qquad (12)$$

From the initial conditions $x(0) = a_0, \dot{x}(0) = 0$ concerning the solution it results

$$c = \frac{1}{a_0^2}, \omega^2 = ck \qquad (13)$$

and than

$$x(t) = \frac{a_0}{\sqrt{2kt + 1}}(\cos t + k \sin t) \qquad (14)$$

From the relations

$$\omega = \frac{3\mu}{8}\left(k^2 + 1\right), \omega^2 = ck \;\; and \;\; k \in R$$

we obtain a condition about the low parameter

$$\mu \le \frac{4}{3} a_0^2 \qquad (15)$$

3. Conclusion

The presented method permits the solving of the non-linear mechanical systems and was chosen as example a system with nonlinear damping forces. The method presents the solving and the evaluation of the value of the low positive parameter in order to make the best approximation.

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