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CONSIDRATIONS UPON THE STUDY OF A NON-LINEAR DYNAMIC SYSTEM FOR VARIOUS OPERATING CONDITIONS

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ABSTRACT

A dynamic system that models the fixing of a fan is studied, by means of the non-linear vibration theory as well as of the dynamic system theory. Three different operation conditions are investigated: autonomous system, non-autonomous system subject to a harmonic excitation and non-autonomous system subject the simultaneous action of two harmonic excitations.

1.PRESENTATION OF THE MODEL

The paper studies a dynamic system , consisting of a mass M, supported in a horizontal plane by three cables, connected to a circular frame (Fig. 1). The cables, of lengths l, Young's moduli E, cross-sectional areas A and negligible masses (with respect to M), are not tensed by any external forces. The system can model the fixing of a fan.

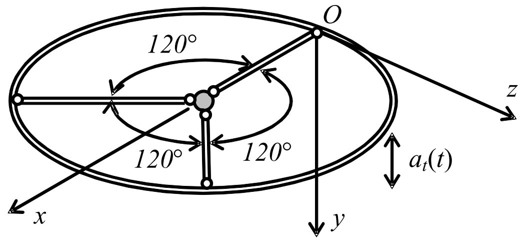

Fig. 1. The studied model

By using some methods of the non-linear vibration theory, as well as of the dynamic system theory, this single-degree-of-freedom system is studied in three different operation conditions:

- as an autonomous one, if the frame is considered fixed;
- as a non-autonomous one, if the frame is subject to vertical vibrations, determined by the acceleration $a_t(t)$ produced by a harmonic excitation;
- as a non-autonomous one, if the frame is subject to vertical vibrations, determined by the acceleration $a_t(t)$ produced by the simultaneous action of two harmonic excitations.

By determining and analysing the equation of motion, the behavior of the system is investigated for certain operation conditions, which makes it possible to take technical measures in order to avoid risk domains.

2.THE EQUATION OF MOTION

The equation of motion can be determined by using Lagrange equations of the second species.

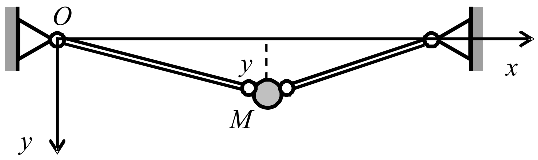

Fig. 2. Vertical projection of the studied model

For the autonomous system (Fig. 2), which is conservative, the Lagrange equation takes the form

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{y}}\right)-\frac{\partial E}{\partial y}=\frac{\partial U}{\partial y}, \tag{0}$$

where the kinetic energy has been introduced,

$$E = \frac{1}{2} M\dot{y}^2 , \qquad (0)$$

as well as the force function,

$$U = -\frac{3EA}{2l}\left(\sqrt{l^2 + y^2} - l_0\right)^2 . \qquad (0)$$

In the previous formula, l_0 is the length of the untensed cable.

By replacing (0) and (0) in (0), the differential equation of motion of the autonomous system is obtained:

$$M\ddot{y} + \frac{3EA}{l}\left(1 - \frac{l_0}{\sqrt{l^2 + y^2}}\right) y = 0 . \qquad (0)$$

If the non-autonomous system is considered, the differential equation of motion becomes:

$$M\ddot{y} + \frac{3EA}{l}\left(1 - \frac{l_0}{\sqrt{l^2 + y^2}}\right) y = Ma_t(t) . \qquad (0)$$

For small elongations (small values of parameter y), the approximation

$$\left(1 + \frac{y^2}{l^2}\right)^{-\frac{1}{2}} = 1 - \frac{\frac{1}{2}}{1!}\frac{y^2}{l^2} + \ldots \cong 1 - \frac{y^2}{2l^2} \qquad (0)$$

can be made, which allows equation (0) to be rewritten as

$$\ddot{y} + \frac{3EA}{Ml}\left(1 - \frac{l_0}{l}\right) y + \frac{3EAl_0}{2Ml^4} y^3 = a_t(t) \qquad (0)$$

or

$$\ddot{y} + \omega^2 y + \mu y^3 = a_t(t) , \qquad (0)$$

where the following notations have been introduced:

$$\begin{cases} \omega^2 = \dfrac{3EA}{Ml}\left(1 - \dfrac{l_0}{l}\right) \\ \mu = \dfrac{3EAl_0}{2Ml^4} . \end{cases} \qquad (0)$$

It can be seen from (0) that the system has the non-linear characteristic (Fig. 3)

$$f(y) = \omega^2 y + \mu y^3 , \qquad (0)$$

which is a strong one, since

$$f''(y) = 6\mu y = \frac{9EAl_0}{Ml^4} y > 0 \quad \text{if} \quad y > 0 . \qquad (0)$$

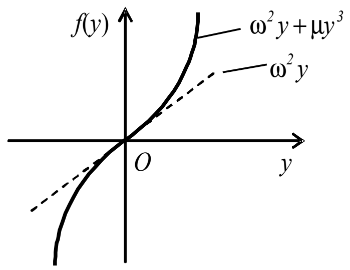

Fig. 3. Strong non-linear characteristic

3.ANALYSIS OF THE SYSTEM

In the following, the cases of the autonomous and non-autonomous system, respectively, are analysed.

3.1.The autonomous system

In this case, the differential equation of motion (0) becomes

$$\ddot{y} + \omega^2 y + \mu y^3 = 0 \qquad (0)$$

and its periodical solution (with the circular frequency Ω) will be found by using Linstadt method .

The method can be applied if parameter μ is small and it consists in rewriting the equation as

$$\Omega^2 \frac{d^2 y}{d\tau^2} + \omega^2 y + \mu f_1(y) = 0 , \qquad (0)$$

where

$$\tau = \Omega t , \qquad (0)$$

and in expanding the solution and Ω^2 as power series of μ:

$$y(\tau) = y_0(\tau) + \mu y_1(\tau) + \mu^2 y_2(\tau) + \ldots \qquad (0)$$

$$\Omega^2 = a_0 + \mu a_1 + \mu^2 a_2 + \ldots \qquad (0)$$

By replacing (0) and (0) in (0) and by equating with zero the coefficients of the powers of μ, a system of second order linear differential equations with the unknowns $y_0(\tau)$, $y_1(\tau)$, $y_2(\tau)$,… is obtained:

$$\begin{cases} a_0 \dfrac{d^2 y_0}{d\tau^2} + \omega^2 y_0 = 0 \\ a_0 \dfrac{d^2 y_1}{d\tau^2} + \omega^2 y_1 = -y_0^3 - a_1 \dfrac{d^2 y_0}{d\tau^2} \\ a_0 \dfrac{d^2 y_2}{d\tau^2} + \omega^2 y_2 = -3y_0^2 y_1 - a_1 \dfrac{d^2 y_1}{d\tau^2} - a_2 \dfrac{d^2 y_0}{d\tau^2} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases} \qquad (0)$$

For $\mu = 0$, the case of linear vibrations is obtained, so that

$$a_0 = \omega^2 , \qquad (0)$$

while the solution which satisfies the initial conditions

$$t = 0 \quad \Rightarrow \quad \begin{cases} y = a \\ \dot{y} = 0 \end{cases} \qquad (0)$$

is

$$y_0(\tau) = a \cos \tau . \qquad (0)$$

By replacing this result, the second equation of the system (0) becomes successively

$$\omega^2 \frac{d^2 y_1}{d\tau^2} + \omega^2 y_1 = -a^3 \cos^3 \tau + a_1 a \cos \tau , \quad (0)$$

$$\frac{d^2 y_1}{d\tau^2} + y_1 = -\frac{a^3}{4\omega^2} \cos 3\tau + \left(\frac{a_1 a}{\omega^2} - \frac{3a^3}{4\omega^2} \right) \cos \tau . \quad (0)$$

By equating with zero the coefficient of *cos* τ, it follows

$$a_1 = \frac{3a^2}{4} , \quad (0)$$

$$\frac{d^2 y_1}{d\tau^2} + y_1 = -\frac{a^3}{4\omega^2} \cos 3\tau . \quad (0)$$

The solution for which initial conditions (0) remain unchanged is

$$y_1(\tau) = \frac{a^3}{32\omega^2} (\cos 3\tau - \cos \tau) , \quad (0)$$

hence the first order approximations for (0) and (0) are

$$\begin{aligned} y(\tau) &\cong y_0(\tau) + \mu y_1(\tau) = \\ &= \left(a - \frac{\mu a^3}{32\omega^2} \right) \cos \tau + \frac{\mu a^3}{32\omega^2} \cos 3\tau , \end{aligned} \quad (0)$$

$$\Omega \cong \sqrt{a_0 + \mu a_1} = \omega \sqrt{1 + \frac{3a^2}{4\omega^2} \mu} . \quad (0)$$

By similar procedures, from the third equation of the system (0), rewritten successively as

$$\begin{aligned} \omega^2 \frac{d^2 y_2}{d\tau^2} &+ \omega^2 y_2 = \\ &= -\frac{3a^5}{32\omega^2} (\cos 3\tau - \cos \tau) \cos^2 \tau + \\ &+ \frac{3a^5}{128\omega^2} (9 \cos 3\tau - \cos \tau) + a_2 a \cos \tau , \end{aligned} \quad (0)$$

$$\begin{aligned} \frac{d^2 y_2}{d\tau^2} + y_2 &= -\frac{3a^5}{128\omega^4} \cos 5\tau + \\ &+ \frac{3a^5}{16\omega^4} \cos 3\tau + \left(\frac{3a^5}{128\omega^4} + \frac{a_2 a}{\omega^2} \right) \cos \tau , \end{aligned} \quad (0)$$

the following results are obtained:

$$a_2 = -\frac{3a^4}{128\omega^2} , \quad (0)$$

$$\frac{d^2 y_2}{d\tau^2} + y_2 = -\frac{3a^5}{128\omega^4} \cos 5\tau + \frac{3a^5}{16\omega^4} \cos 3\tau , \quad (0)$$

$$\begin{aligned} y_2(\tau) &= \frac{23a^5}{1024\omega^4} \cos \tau - \frac{3a^5}{128\omega^4} \cos 3\tau + \\ &+ \frac{a^5}{1024\omega^4} \cos 5\tau . \end{aligned} \quad (0)$$

The second order approximations for (0) and (0) are

$$\begin{aligned} y(\tau) &\cong y_0(\tau) + \mu y_1(\tau) + \mu^2 y_2(\tau) = \\ &= \left(a - \frac{\mu a^3}{32\omega^2} + \frac{23\mu^2 a^5}{1024\omega^4} \right) \cos \tau + \\ &+ \left(\frac{\mu a^3}{32\omega^2} - \frac{3\mu^2 a^5}{128\omega^4} \right) \cos 3\tau + \frac{\mu^2 a^5}{1024\omega^4} \cos 5\tau , \end{aligned} \quad (0)$$

$$\begin{aligned} \Omega &\cong \sqrt{a_0 + \mu a_1 + \mu^2 a_2} = \\ &= \omega \sqrt{1 + \frac{3a^2}{4\omega^2} \mu - \frac{3a^4}{128\omega^2} \mu^2} . \end{aligned} \quad (0)$$

Equation (0) is equivalent with the first order differential system

$$\begin{cases} \dot{y} = v \\ \dot{v} = -\omega^2 y - \mu y^3 . \end{cases} \quad (0)$$

By adding the equations of this system, multiplied with $\omega^2 y$ and v, respectively, the following equality is obtained

$$v\dot{v} + \omega^2 y \dot{y} + \mu y^3 \dot{y} = 0 . \quad (0)$$

This relation can be integrated:

$$\frac{v^2}{2} + \omega^2 \frac{y^2}{2} + \mu \frac{y^4}{4} = C . \quad (0)$$

The last equation describes the trajectories followed by the system in the phase plane, represented in Figure 4, for ω=*1*, μ=*1* and *C*=*1*, *2*, *3*, *4*.

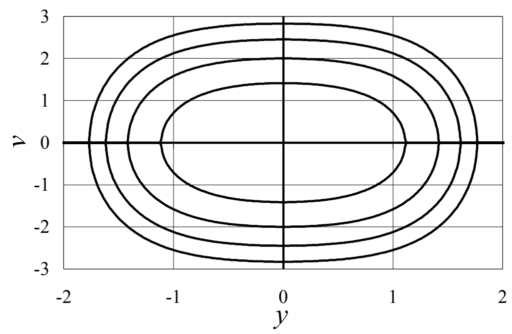

Fig. 4. Phase plane

3.2. The non-autonomous system under the action of one harmonic excitation

If the system is subject to a harmonic excitation, the equation of motion (0) becomes

$$\ddot{y} + \omega^2 y + \mu y^3 = q_0 \cos \Omega t . \quad (0)$$

An approximate periodical solution with the same frequency as the perturbation force is seeked.

Considering a trigonometrical series expansion containing only the odd multiples of Ωt , the approximate solution will be:

$$y(\tau) \cong A_1 \cos \Omega t + A_3 \cos 3\Omega t . \quad (0)$$

By replacing (0) in (0), by neglecting the terms containing $A_1^2 A_3$, $A_1 A_3^2$, A_3^3 and by equating with zero the coefficients of the trigonometrical functions, the following system is obtained:

$$\begin{cases}\left(\omega^2-\Omega^2\right)A_1+\dfrac{3}{4}\mu A_1^3-q_0=0\\ \left(\omega^2-9\Omega^2\right)A_3+\dfrac{1}{4}\mu A_1^3=0.\end{cases} \qquad (0)$$

Assuming that the order of μ and q_0 is $0(\varepsilon)$, where ε is a small parameter and admitting that $\omega^2-\Omega^2=0(\varepsilon)$, it results from (0):

$$A_3=\frac{1}{32\omega^2}\mu A_1^3+0\left(\varepsilon^2\right). \qquad (0)$$

It follows from (0) the approximate solution

$$y(\tau)\cong A_1\cos\Omega t+\frac{1}{32\omega^2}\mu A_1^3\cos 3\Omega+0\left(\varepsilon^2\right). \qquad (0)$$

The coefficient A_1 of the first harmonic can be determined from (0), by writing

$$\Omega^2\cong\omega^2+\frac{3}{4}\mu A_1^2-\frac{q_0}{A_1}+0\left(\varepsilon^2\right) \qquad (0)$$

or, by using $A_1=\pm A$

$$\Omega^2\cong\omega^2+\frac{3}{4}\mu A^2\mp\frac{q_0}{A}+0\left(\varepsilon^2\right). \qquad (0)$$

For $q_0=const$, the graph of the function $A_1\left(\dfrac{\Omega^2}{\omega^2},A\right)$ is called *resonance curve*.

3.3.The non-autonomous system under the simultaneous action of two harmonic excitations

If the system is simultaneously acted by two harmonic excitations with not equal frequencies, the equation of motion (0) becomes

$$\ddot y+\omega^2y+\mu y^3=q_1\cos\Omega_1t+q_2\cos\Omega_2t. \qquad (0)$$

Since for $\mu=0$ the superposition of the effects takes place, an approximate solution is seeked of the form

$$y(\tau)\cong A_1\cos\Omega_1t+A_2\cos\Omega_2t+\mu u(t), \qquad (0)$$

where $\mu u(t)$ is a correction term due to the elastic non-linearity.

By replacing (0) in (0) and by equating with zero the coefficients of the trigonometrical functions, a system with the unknowns A_1, A_2 and $u(t)$ is obtained. If the powers of μ are neglected, this system is

$$\begin{cases}\omega^2-\Omega_1^2+\dfrac{3}{4}\mu\left(A_1^2+2A_2^2\right)-\dfrac{q_1}{A_1}=0\\ \omega^2-\Omega_2^2+\dfrac{3}{4}\mu\left(2A_1^2+A_2^2\right)-\dfrac{q_2}{A_2}=0\\ \ddot u+\omega^2u=-\dfrac{1}{4}\left\{A_1^3\cos 3\Omega_1t+\right.\\ \quad+3A_1^2A_2\left[\cos\left(2\Omega_1+\Omega_2\right)t+\cos\left(2\Omega_1-\Omega_2\right)t\right]+\\ \quad+3A_1A_2^2\left[\cos\left(\Omega_1+2\Omega_2\right)t+\cos\left(\Omega_1-2\Omega_2\right)t\right]+\\ \quad\left.+A_2^3\cos 3\Omega_2t\right\}.\end{cases} \qquad (0)$$

The last equality is fulfilled for

$$\begin{aligned}u(t)=-\frac{1}{4}\Bigg[&A_1^3\frac{\cos 3\Omega_1t}{\omega^2-9\Omega_1^2}+\\ &+3A_1^2A_2\frac{\cos\left(2\Omega_1+\Omega_2\right)t}{\omega^2-\left(2\Omega_1+\Omega_2\right)^2}+\\ &+3A_1^2A_2\frac{\cos\left(2\Omega_1-\Omega_2\right)t}{\omega^2-\left(2\Omega_1-\Omega_2\right)^2}+\\ &+3A_1A_2^2\frac{\cos\left(\Omega_1+2\Omega_2\right)t}{\omega^2-\left(\Omega_1+2\Omega_2\right)^2}+\\ &+3A_1A_2^2\frac{\cos\left(\Omega_1-2\Omega_2\right)t}{\omega^2-\left(\Omega_1-2\Omega_2\right)^2}+\\ &+A_2^3\frac{\cos 3\Omega_2t}{\omega^2-9\Omega_2^2}\Bigg].\end{aligned} \qquad (0)$$

From the analysis of the results, it can be seen that the two excitation forces generate forced vibrations with the same frequencies as theirs, but also supplementary vibrations, with the frequencies $3\Omega_1$ $2\Omega_1+\Omega_2$, $2\Omega_1-\Omega_2$, $\Omega_1+2\Omega_2$, $\Omega_1-2\Omega_2$, $3\Omega_2$.

4.CONCLUSIONS

By determining and analysing the differential equation of motion, the behavior of the dynamical system has been studied under different assumptions: autonomous system, non-autonomous system under the action of one harmonic excitation and non-autonomous system under the simultaneous action of two harmonic excitations with not equal frequencies.

Approximate periodical solutions obtained in each situation provide a theoretical tool for measures of avoiding risk domains in certain technical applications.

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