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CALCULATION METHOD OF STRAIGHT UNSUPPORTED (FREELY DEFORMABLE) BARS SUBJECTED TO IN-PLANE BENDING STRESS

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ABSTRACT

This paper describes a method of determining the cross section arrows and rotations of a straight unsupported bar subjected to bending stress. The straight bar is in a state of static balance while subjected to external loads and not resting on any external point. The calculation relations obtained enable the specialists to determine both the rotation and the displacement of all the cross sections of a straight unsupported bar, which occur further from the external in-plane bending loading. Please note that these displacements are calculated against the initial position of the straight bar. When the exterior bending stress starts to be higher than zero, the unsupported bar is in a continuous static balance and it bends continuously until the exterior load reaches its highest value. According to the method described hereunder, the arrow and rotation are calculated for this last bar position. The same method is also employed to determine the positions of the cross sections that do not undergo any displacement from their initial position. Along the whole straight bar, these sections are at least two.

KEYWORDS: arrow, rotation, displacement, free deformation, in-plane bending, imaginary load

1. INTRODUCTION

When subjected to in-plane bending stresses, an originally straight bar bends and its geometrical axis becomes a plane curve having the shape of an arc of a circle. The calculation of the displacements encompassed by the cross sections of such a bar is designed to establish the deformed shape of the bar's geometrical axis.

During in-plane bending loading, the cross sections of a straight bar are displaced and rotated against their original position before the stress [1]. Bernoulli's hypothesis applies to elastic bending loading, according to which a plane section perpendicular to the bar axis before the stress (deformation) remains plane and perpendicular to the bar axis throughout the application of the load [2]. The new position of the bar cross section is defined by two parameters:

- the $v(x)$ arrow, which is the vertical displacement of the center of gravity of the cross section, fig. 1;
- the $\varphi(x)$ rotation, which is the angle formed by the bar cross section before bending and the bar cross section after bending, fig. 1.

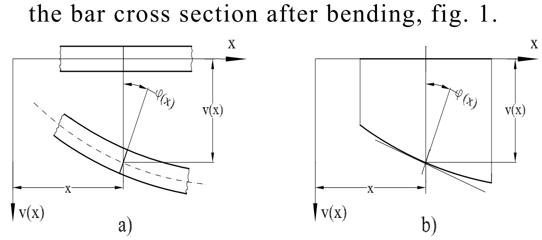

Fig. 1. v(x) arrow and φ(x) rotation; bar subjected to in-plane bending; b) deformation of the bar geometrical axis

It is well known that between the $v(x)$ arrow and $\varphi(x)$ rotation of the cross sections of a straight bar subjected to in-plane bending there is the following differential ratio [3]:

$$\frac{M(x)}{EI_z} = \frac{d\varphi(x)}{dx} = \frac{d^2v(x)}{dx^2} \tag{1}$$

where:

- M(x) is the expression of the bending moment for the same x cross section for which the v(x) arrow and φ(x) rotation are calculated;
- E is Young’s modulus;
- I_z is the axial moment of inertia calculated for the same x section, beyond the z axis, which is perpendicular to the [x, v(x)] plane in fig. 1.

The first integration of the $\left[M(x)/EIz\right]$ ratio results into the φ(x) cross section rotation expression, while the second integration of this expression leads to the determination of the ratio concerning the v(x) cross section arrow. The expression for the M(x) moment is similar to that for a bar section where no changes in either the exterior load variation law or the cross section shape variation law occur. In order to determine the deformed shape of the whole straight bar, we have to find the M(x) moment variation law expressions for each bar section. If we apply the two integrations to the left component of the ratio (1), two integration constants will result for each bar section. If the bar has **n** sections, we will get **2n** integration constants. The latter will be determined using (2n-2) continuity requirements applied when changing sections and two bar support requirements. Please note that, without the two support requirements, and using only the double integration of the left ratio component (1) for each bar section, we will not be able to determine the arrow and rotation expressions of the cross section of a straight bar subjected to in-plane bending stress.

2. STRAIGHT UNSUPPORTED BAR DEFORMATION

When a straight bar is deformation-free and unsupported, the in-plane bending stress should be able to provide the self-balance at any point in time when its value increases from zero to the highest value [4]. Throughout this increase, the bar bends but it also preserves its static balance under the exterior load applied to it. In practice, there are cases when a straight bar subjected to in-plane bending is not actually supported and still preserves its balance due to the balance of the exterior load. However, the problem in this case is that the two *support requirements* can no longer apply to determine the expressions of the cross section arrows and rotations. This means that the system of **n** equations with **n** unknown integration constants is no longer possible.

Further more in the examination of a bar section which is dx long, we may conclude that the following differential relations may be established between the q(x) exterior load, the stress as shear force applied to the x bar section and the stress as bending moment applied to the same bar section [3]:

$$q(x) = \frac{dT(x)}{dx} = \frac{d^2M(x)}{dx^2} \tag{2}$$

Note that there is obviously a similarity between these differential relations and the relations (1). Consequently, if for a straight bar subjected to in-plane bending we draw a M(x) moment diagram on the whole bar length, the area of that diagram may become a new $q^*(x)$ load applied to that bar. If we consider the analogy above between the relations (1) and (2), the x cross section arrow and rotation expressions are:

$$\begin{aligned} EI_z\varphi(x) &= T^*(x) + C_1 \\ EI_z v(x) &= M^*(x) + C_1 x + C_2 \end{aligned} \tag{3}$$

where:

- $T^*(x)$ and $M^*(x)$ are the x section stress as shear force and stress as bending moment, respectively, determined considering $q^*(x)=M(x)$ as being an *imaginary load* applied to the straight bar;
- C_1 and C_2 are the integration constants that should be determined according to the connection and continuity requirements.

The validity requirement for ratio (3) is that the *real beam* in fig. 2a turns into an *imaginary beam*, fig. 2b. Thus, the free ends of the real beam for which φ≠0 and v≠0 become embedded ends when the beam is imaginary, which means that $T^*\neq0$ and $M^*\neq0$. Therefore, the support requirements are automatically fulfilled, meaning that $C_1=C_2=0$, and a connection may be established between the rotation of the real bar φ(x) section and the effort as imaginary bar $T^*(x)$ shear force, as well as between the real bar v(x) section arrow and the effort as imaginary bar $M^*(x)$ bending moment. It is common knowledge that, if we are able to determine the arrow and rotation of each unsupported bar section subjected to bending, we will also be able to determine the arrow and rotation expressions at any other point on the bar. We will reveal the relations allowing this calculation further on.

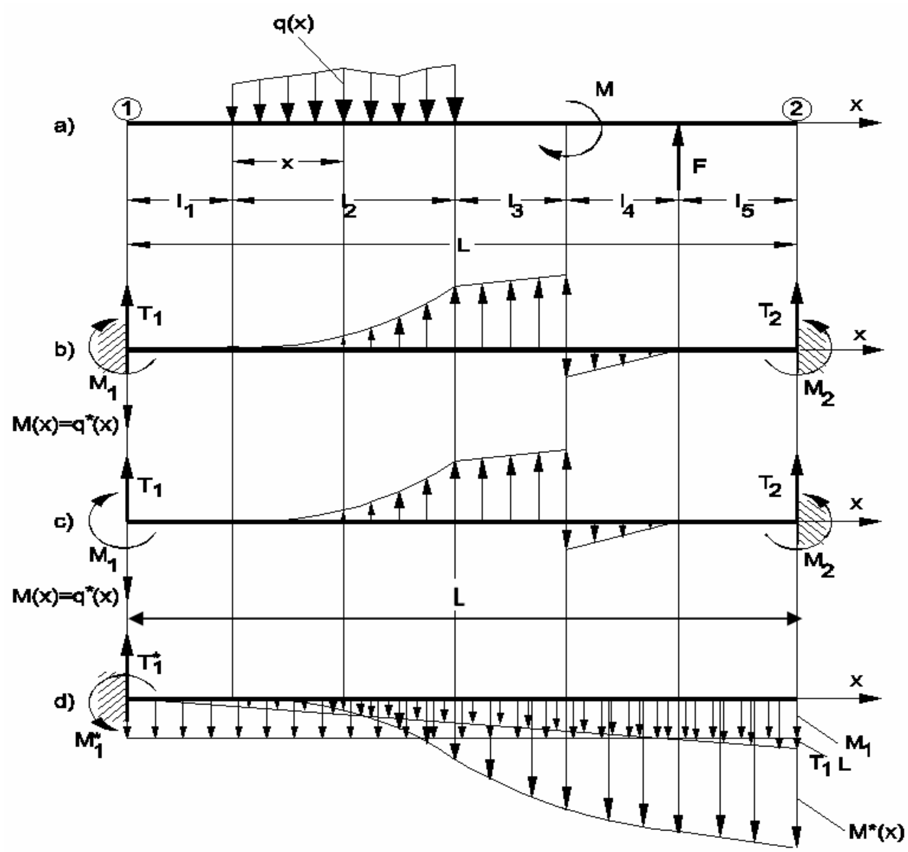

Fig. 2. Rotation and arrow determination work stages applied to the first end of the unsupported bar

Considering the data above, the φ_1 arrow and v_1 rotation at the first end of the real unsupported bar, fig. 2a, may be determined using the relations:

$$\varphi_1 = \frac{T_1}{EI_z} \qquad v_1 = \frac{M_1}{EI_z} \tag{4}$$

In the equation system provided by ratio (4), T_1 and M_1 are the reactions (shear force and bending moment, respectively) in the embedding at the first end of the imaginary bar, fig. 2b, where the load, $q^*(x) = M(x)$, is the moment diagram area calculated for the real unsupported beam, fig. 2a. Consequently, if we find a method to determine the T_1, M_1, T_2 and M_2 reactions in the imaginary bar with both embedded ends, fig. 2b, we will be able to determine the unsupported real bar arrow and rotation, fig. 2a.

We should bear in mind, however, that the lower part of the axis x should be considered the positive part of the imaginary $q^*(x)$ load, in order to achieve a match between the signs of the displacements and those of the reactions ($\varphi_1 - T_1$ and $v_1 - M_1$), as shown in fig. 2b.

3. DETERMINATION OF ARROW AND ROTATION CALCULATION RELATIONS AT THE FIRST END OF THE UNSUPPORTED BAR

The straight bar subjected to bending in fig. 2b is a double statically undetermined bar [5]. This bar may turn into a ”statically determined” one if we remove the support on which the first end rests and we keep the T_1 and M_1 reactions as a shear force and a bending moment acting on the free end of the bar, fig. 2c. The bar in fig. 2c may be the equivalent of the bar in fig. 2b provided that the φ_1^* rotation and v_1^* arrow at the first end of the bar are zero. The φ_1^* rotation and v_1^* arrow expressions occurring at the first end of the bar are similar to those in equation (4) that is:

$$\varphi_1^* = \frac{T_1^*}{EI_z}$$
$$v_1^* = \frac{M_1^*}{EI_z} \qquad (5)$$

where T_1^* and M_1^* are the stress as shear force and stress as bending moment, respectively, at the first end of the imaginary bar in fig. 2b, determined considering $q^*(x)=M(x)$ as being an *imaginary load* applied to the same bar.

Fig. 2d shows the separate moment diagrams resulted the loading of the bar in fig. 2c with the imaginary $q^*(x)=M(x)$ load, with the T_1 shear force and with the moment M_1 bending moment. The area of this diagram may be considered as an imaginary load acting on the bar in fig. 2d, whose first end is attached, in order to continue meeting the $\varphi_1^* = 0$ and $v_1^* = 0$ requirements. Thus, the T_1^* and M_1^* reactions will result from the static balance equations of the bar in fig. 2d:

$$T_1^* = T_1\frac{L^2}{2} + M_1 L + \sum_i \left(A_i^*\right)$$
$$M_1^* = T_1\frac{L^3}{3} + M_1\frac{L^2}{2} + \sum_i \left(A_i^* x_{G_i-1}^*\right) \qquad (6)$$

where:

- L is the full length of the bar;
- A_i^* is the $M^*(x)$ moment diagram area, *considered to be the sign resulting from calculations*, for each bar section;
- x^*_{Gi-1} is the distance from the center of gravity of each A_i^* area in point 1.

The last two values may be calculated according to the relations:

$$\begin{cases} A_i^* = \int_0^{l_i}\left(M^*(x_i)dx_i\right) \\ x_{G_i-1}^* = \sum_{i-1} l_{i-1} + \dfrac{\int_0^{l_i}\left(M^*(x_i)\bullet x_i \bullet dx_i\right)}{\int_0^{l_i}\left(M^*(x_i)dx_i\right)} \end{cases} \qquad (7)$$

where:

- $M^*(x_i)$ is the bending moment expression for each **i** bar area, provided by the imaginary load $q^*(x)=M(x)$;
- l_i is the length of each **i** bar area where the $M^*(x_i)$ moment expression is valid.

Therefore, if we use ratio (5), the first bar end section arrow and rotation in fig. 2c are:

$$\varphi_1^* = \frac{T_1^*}{EI_z} = \frac{1}{EI_z}\left[\ T_1\frac{L^2}{2} + M_1 L + \sum_i \left(A_i^*\right)\ \right] = 0$$
$$v_1^* = \frac{M_1^*}{EI_z} = \frac{1}{EI_z}\left[\ T_1\frac{L^3}{3} + M_1\frac{L^2}{2} + \sum_i \left(A_i^* x_{G_i-1}^*\right)\ \right] = 0 \qquad (8)$$

As already said, in order to preserve the similarities between the bar in fig. 2c and that in fig. 2b, the φ_1^* rotation and v_1^* arrow at the first end of the bar should be zero. Thus, if we solve the equation system in ratio (7) and if we use the relations (4), we get the following φ_1 rotation and v_1 arrow calculation relations, which are actually the **calculation of the displacement of the first free end of the real bar** in fig. 2a:

$$\varphi_1 = \frac{T_1}{EI_z} = \frac{1}{EI_z}\left[\ \frac{6}{L^2}\sum_i \left(A_i^*\right) - \frac{12}{L^3}\sum_i \left(A_i^* x_{G_i-1}^*\right)\ \right]$$
$$v_1 = \frac{M_1}{EI_z} = \frac{1}{EI_z}\left[\ \frac{6}{L^2}\sum_i \left(A_i^* x_{G_i-1}^*\right) - \frac{4}{L}\sum_i \left(A_i^*\right)\ \right] \qquad (9)$$

Once we determined the rotation and arrow at the first free bar end, using the expressions provided by ratio (9), we may also determine, relying on the continuity requirements, the rotations and arrows of any real unsupported bar section. The calculation relations that may be employed to determine them are:

$$\varphi(x_i) = \varphi_1 - \frac{A_{1-x_i}}{EI_z}$$
$$v(x_i) = v_1 + \varphi_1 x_i - \frac{S_{1-x_i}}{EI_z} \qquad (10)$$

where:

- φ_1 and v_1 are the rotation and arrow, respectively, at the first unsupported bar end, and they are determined using the relations (9);
- x_i is the distance from the examined section at the first bar end;
- A_{1-x_i} is the $M(x)=q^*(x)$ moment diagram area, considered with its respective sign and from the first end to the x_i section;
- S_{1-x_i} is the static moment of the A_{1-x_i} area (considered with the sign) calculated in relation to the vertical axis crossing the x_i section.

Using the relations (9) and the relations (10), we may therefore be able to draw the variation diagram along the unsupported bar, for the rotations and arrows of the cross sections of that bar subjected to in-plane bending.

Fig. 3 shows a practical example and the work stages required to draw those rotation and arrow variations.

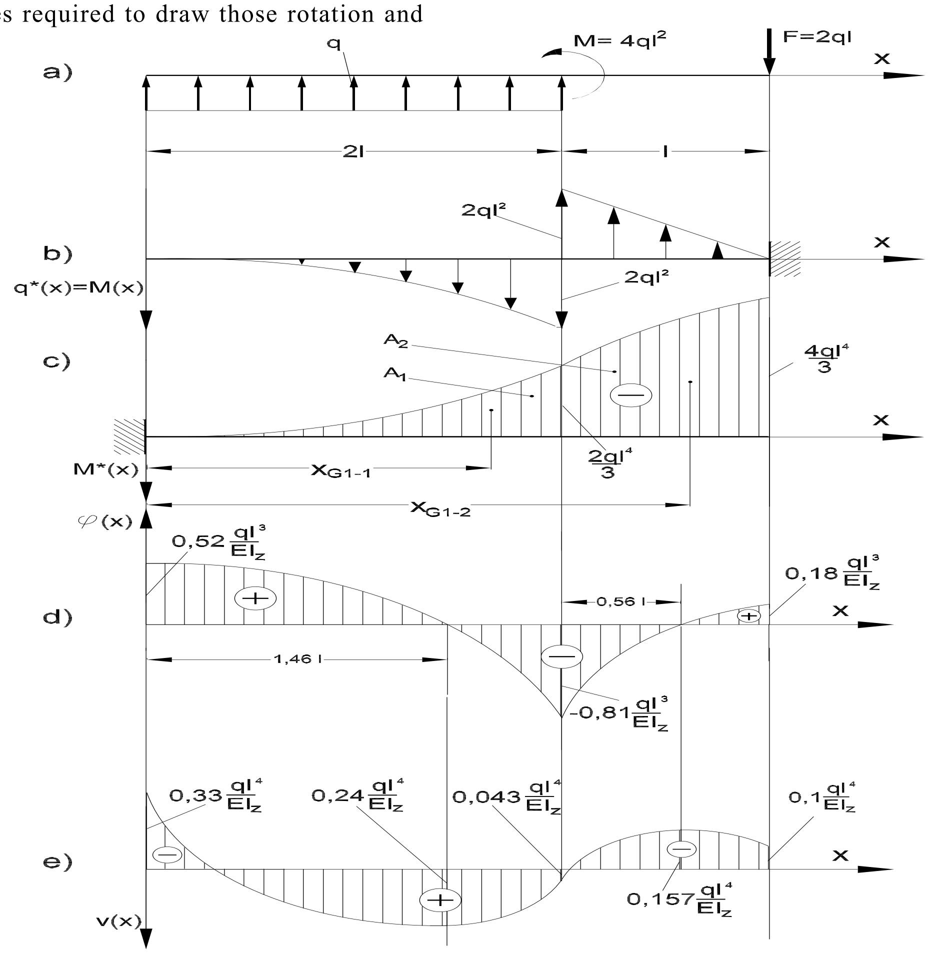

Fig. 3. Rotation and arrow variation diagram of an unsupported bar subjected to in-plane bending

According to fig. 3a, the straight bar subjected to in-plane bending is unsupported and it is in a state of static balance due solely to the exterior load acting on it:

$$\begin{aligned} &\sum_i T_i = q \cdot 2l - 2ql = 0 \\ &\sum_i M_i = 2ql \cdot 3l - 4ql^2 - q \cdot 2l \cdot l = 0 \end{aligned} \tag{11}$$

Fig. 3b shows the $q^*(x)=M(x)$ imaginary load applied to a bar whose left end is attached, which is similar to the bar in fig. 2c. According to ratio (9), note that it is not necessary to also represent the T_1 and M_1 reactions; this may only be determined using the moment diagram area provided by the imaginary load. Following the steps described above, we may draw the $M^*(x)$ moment diagram, fig. 3c, for the bar whose left end is attached, just like in fig. 2d. Using the A_1 and A_2 areas in this diagram, as well as the relations (9), we may calculate the φ_1 rotation and v_1 arrow for the left bar end:

$$\begin{aligned} \varphi_1 &= 0.52 \frac{ql^3}{EI_z} \\ v_1 &= 0.33 \frac{ql^4}{EI_z} \end{aligned} \tag{12}$$

Once these values are determined and using the relations (10), we may draw the $\varphi(x)$ rotation and $v(x)$ arrow variation diagrams along the whole straight unsupported bar, which is in a state of static balance due solely to the exterior load acting on it as in-plane bending, fig.s 3d and 3e. Figure 3d shows that there are two bar sections that will not undergoany displacement throughout stress increase from zero to the highest value. Thus, next to these sections, the arrow is zero and the rotation is at its peak.

It is very important to specify that the straight unsupported bars, which are in a state of permanent static balance throughout in-plane bending stress increase from zero to the highest value, have at least two sections that will not undrego any displacement against their original position. In another paper, we showed that for straight unsupported bars subjected to axial loading there is at least one section that undergoes displacement (rotation) against its original position.

4. CONCLUSIONS

There are not many actual cases of unsupported bars in static balance subjected to the action of different bending stresses. When the bar starts to be subjected to a bending stress, the exterior loads increase from zero to the highest value and they are in a permanent static balance. It is obvious that the balance of the deformed neutral fiber should be achieved when the load reaches its peak [5].

Using the relations (9) and the relations (7), one may determine the φ_1 rotation and v_1 arrow at the **first** free end of the straight unsupported bar subjected to in-plane bending. The following work stages should be covered in order to achieve that:

- draw the $q^*(x)=M(x)$ bending moment diagram due to the exterior loads applied to the unsupported bar;
- consider $q^*(x)$ as imaginary load for a new bar whose second end is embedded, while the load acts from the bar;
- draw the $M_i^*(x)$ moment diagram for this bar;
- use the relations (7) and (9) to determine the φ_1 rotation and v_1 arrow at the first free end of the straight unsupported bar, where you should consider the resulted sign of those bending moments;
- use the relations (10) to draw the rotation and arrow variation diagram on the whole L length of the bar, where you should also consider the resulted sign for the $M_i(x)$ diagram.

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