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MODAL CALCULUS OF THE REINFORCED CONCRETE BRIDGES MODELED AS A RIGID SOLID BEARED ON VISCOUS ELASTIC NEOPRENE SUPPORTS

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ABSTRACT

This paper proposes an approach of a six degree dynamic model of a rigid-solid with some types of symmetries. These symmetries lead to simplified mathematical models, which are more easily to solve. If the rigid-solid is symmetrical beared by triorthogonal elastic links, the mathematical model becomes still simple and the vibrations are decoupled into four subsystems of movements: side slipping and rolling, forward motion and pitching, lifting motion, gyration. There are two case studies of modal analysis: for a viaduct with five arches made from reinforced concrete „U” beam and for an arch (between two piers of the viaduct) made from four reinforced concrete „U” beams.

KEYWORDS: modal calculus, eigenvalues, concrete bridges vibrations

1. INTRODUCTION

The mathematical modeling uses the physical model of the rigid solid with six degrees of freedom (6DOF) with a finite number of viscous-elastic bearings [2], [3], [6], [11]. Dimensional and inertial characteristics of the rigid solid and rheological characteristics of the bearings (stiffness and damping) can be experimentally determined by direct measurements and by static and/or dynamic testing. According to [1], [4], [5], [7] and [8], the differential equations of the movements of the rigid solid with viscous-elastic bearings are coupled by stiffness and damping coefficients. The system of the equations can be written as follows:

$$\underline{\underline{A}}\ddot{\underline{\underline{q}}} + \underline{\underline{B}}\dot{\underline{\underline{q}}} + \underline{\underline{C}}\underline{\underline{q}} = \underline{\underline{f}}\ , \tag{1}$$

where $\underline{\underline{A}}$ is the inertia matrix;

$\underline{\underline{B}}$ is the viscous damping matrix (damping coefficients);

$\underline{\underline{C}}$ is the elasticity matrix (stiffness coefficients);

$\underline{\underline{q}} / \dot{\underline{\underline{q}}} / \ddot{\underline{\underline{q}}}$ are generalized displacements / velocities / accelerations vectors;

$\underline{\underline{f}}$ is the generalized forces vector.

If the damping coefficients are small, the differential equations system becomes:

$$\underline{\underline{A}}\ddot{\underline{\underline{q}}} + \underline{\underline{C}}\underline{\underline{q}} = \underline{\underline{f}} \tag{2}$$

Considering the rigid solid no perturbated, the system of differential equations becomes

$$\underline{\underline{A}}\ddot{\underline{\underline{q}}} + \underline{\underline{C}}\underline{\underline{q}} = \underline{\underline{0}}\ , \tag{3}$$

where $\underline{\underline{0}}$ is the null vector).

If the Cartesian coordinates axis system is central and principal, the quadratic 6×6 inertia matrix becomes diagonal

$$\underline{\underline{A}} = DIAG\lfloor m, m, m, J_x, J_y, J_z \rfloor\ , \tag{4}$$

where m is the rigid solid mass and J_x, J_y, J_z are the principal inertia moments.

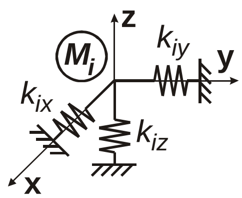

Figure 1 Elastic triorthogonal bearing

2. MODAL ANALYSIS OF THE RIGID SOLID WITH STRUCTURAL SYMMETRIES

Considering that the rigid solid has a vertical axis of symmetry (mass distribution, geometrical configuration, bearings disposal) and the coordinate system is central and principal, the inertia matrix is diagonal. If the elastic bearing system of the rigid solid is composed from n supports with triorthogonal stiffness (k_{ix}, k_{iy}, k_{iz}) like in fig. 1, with the position done by the coordinates $M_i(x_i, y_i, z_i)$ $i = \overline{1,n}$, the elasticity matrix becomes:

$$\underline{C} = \left[\begin{array}{c|c} \underline{C}_{11} & \underline{C}_{12} \\ \hline \underline{C}_{21} & \underline{C}_{22} \end{array}\right], \tag{5}$$

where the component sub matrix have the following structures and coefficients:

$$\underline{C}_{11} = \begin{bmatrix} \sum k_{ix} & 0 & 0 \\ 0 & \sum k_{iy} & 0 \\ 0 & 0 & \sum k_{iz} \end{bmatrix} \tag{6}$$

$$\underline{C}_{12} = \begin{bmatrix} 0 & \sum k_{ix} z_i & 0 \\ -\sum k_{iy} z_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$\underline{C}_{21} = \begin{bmatrix} 0 & -\sum k_{iy} z_i & 0 \\ \sum k_{ix} z_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\underline{C}_{22} = DIAG\Big[\sum\left(k_{iy} z_i^2 + k_{iz} y_i^2\right), \sum\left(k_{iz} x_i^2 + k_{ix} z_i^2\right), \sum\left(k_{ix} y_i^2 + k_{iy} x_i^2\right)\Big] \tag{9}$$

As the inertia matrix is diagonal, the coefficients outside the main diagonal of the elasticity matrix $\underline{C}$ are the coupling terms of the equations of the system (3). Because there are only four non-zero stiffness coefficients ($c_{15} \equiv c_{51}$ and $c_{24} \equiv c_{42}$), the free movements of the rigid solid are decoupled into four subsystems with coupled vibrations. The mathematical models of the subsystems with coupled motion equations are as follows:

a)subsystem (X, φ_y) - side slip movement coupled with rolling movement

$$\begin{cases} m\ddot{X} + X\sum k_{ix} + \varphi_y \sum k_{ix} z_i = 0 \\ J_y \ddot{\varphi}_y + X\sum k_{ix} z_i + \varphi_y \sum\left(k_{iz} x_i^2 + k_{ix} z_i^2\right) = 0 \end{cases} \tag{10}$$

b)subsystem (Y, φ_x) - forward-back movement coupled with pitch movement

$$\begin{cases} m\ddot{Y} + Y\sum k_{iy} - \varphi_x \sum k_{iy} z_i = 0 \\ J_x \ddot{\varphi}_x - Y\sum k_{iy} z_i + \varphi_x \sum\left(k_{iy} z_i^2 + k_{iz} y_i^2\right) = 0 \end{cases} \tag{11}$$

c)subsystem (Z) - up-down movement

$$m\ddot{Z} + Z\sum k_{iz} = 0 \tag{12}$$

d)subsystem (φ_z) - turning movement (gyration)

$$J_z \ddot{\varphi}_z + \varphi_z \sum\left(k_{ix} y_i^2 + k_{iy} x_i^2\right) = 0 \tag{13}$$

In order to determine the natural frequencies and the eigenvalues, we use the next notations:

► for the pulsations of the no coupled movements of translation (along the ccordinate axis)

$$p_X = \sqrt{\frac{\sum k_{ix}}{m}} \tag{14}$$

$$p_Y = \sqrt{\frac{\sum k_{iy}}{m}} \tag{15}$$

$$p_Z = \sqrt{\frac{\sum k_{iz}}{m}} \tag{16}$$

► for the pulsations of the no coupled movements of rotation

$$p_{\varphi_x} = \sqrt{\frac{\sum\left(k_{iy} z_i^2 + k_{iz} y_i^2\right)}{J_x}} \tag{17}$$

$$p_{\varphi_y} = \sqrt{\frac{\sum\left(k_{iz}x_i^2 + k_{ix}z_i^2\right)}{J_y}} \quad (18)$$

$$p_{\varphi_z} = \sqrt{\frac{\sum\left(k_{ix}y_i^2 + k_{iy}x_i^2\right)}{J_z}} \quad (19)$$

►the dynamic coupling terms for the $\left(X,\varphi_y\right)$ and $\left(Y,\varphi_x\right)$ subsystems

$$\begin{cases} \alpha_1 = \dfrac{1}{m}\sum k_{ix}z_i \\ \alpha_2 = \dfrac{1}{J_y}\sum k_{ix}z_i \end{cases} \quad (20)$$

$$\begin{cases} \beta_1 = -\dfrac{1}{m}\sum k_{iy}z_i \\ \beta_2 = -\dfrac{1}{J_x}\sum k_{iy}z_i \end{cases} \quad (21)$$

Considering the relations (14) to (21), the natural pulsations and the eigenvalues of the decoupled subsystems can be determined with the next calculus formulae:

a)for the subsystem $\left(X,\varphi_y\right)$

$$p_{1,2} = SQRT\left\{\frac{1}{2}\left[p_X^2 + p_{\varphi_y}^2 \mp \mp\sqrt{\left(p_X^2 - p_{\varphi_y}^2\right)^2 + 4\alpha_1\alpha_2}\right]\right\} \quad (22)$$

$$\mu_{1,2} = -\frac{1}{2\alpha_1}\left[p_X^2 + p_{\varphi_y}^2 \pm \pm\sqrt{\left(p_X^2 - p_{\varphi_y}^2\right)^2 + 4\alpha_1\alpha_2}\right] \quad (23)$$

b)for the subsystem $\left(Y,\varphi_x\right)$

$$p_{3,4} = SQRT\left\{\frac{1}{2}\left[p_Y^2 + p_{\varphi_x}^2 \mp \mp\sqrt{\left(p_Y^2 - p_{\varphi_x}^2\right)^2 + 4\beta_1\beta_2}\right]\right\} \quad (24)$$

$$\mu_{3,4} = -\frac{1}{2\beta_1}\left[p_Y^2 + p_{\varphi_x}^2 \pm \pm\sqrt{\left(p_Y^2 - p_{\varphi_x}^2\right)^2 + 4\beta_1\beta_2}\right] \quad (25)$$

3. CASE STUDY – MODAL ANALYSIS OF A REINFORCED CONCRETE BRIDGE OF THE ROMANIAN MOTORWAY A3

Figure 2 shows the elevation and the plan view for a bridge made from twenty reinforced concrete beams jointed through a 300 mm thickness reinforced concrete plate. Each beam is beared on the piers and on the abutments of the bridge through four identical viscous-elastic supports made from neoprene; there is a total number of eighty neoprene bearings for the entire bridge. The simplified model of the bridge is shown in fig. 3.

In order to calculate the natural pulsations and frequencies and the eigenvalues of the bridge modeled as in fig. 2, the main characteristics are the following:

■Dimensions (as in detailed engineering drawings and/or measured):

▪for “U” beams: $37100 \times 1700 / 3280 \times 2200$ lenght×width×height [mm]

▪for the bridge: $200000 \times 13300 \times 2500$ lenght×width×height [mm]

■Stiffness of the neoprene bearings (experimental measurements):

$$k_{ix} \equiv k_x = 3{,}15 \times 10^6\ N/m \qquad i = \overline{1{,}80}$$

$$k_{iy} \equiv k_y = 3{,}15 \times 10^6\ N/m \qquad i = \overline{1{,}80}$$

$$k_{iz} \equiv k_z = 650 \times 10^6\ N/m \qquad i = \overline{1{,}80}$$

■Masses and inertia: according to table 1 (calculated)

■Position of the mass center C against the neoprene bearings (calculated): $h = 1454{,}4mm$

■Positions of the neoprene bearings on the viaduct (related to the centered coordinate system Cxyz) as in detailed engineering drawings – see table 2.

Using the relations (14) to (19), the natural pulsations p and the natural frequencies f (calculated acc. to [9] and [10]) of the uncoupled vibrations for the six degrees of dynamic freedom are shown in table 3.

The figures from table 4 show the values of the natural pulsations and frequencies and of the eigenvalues for the decoupled subsystems (with coupled movements) for a bridge section (arche) composed from four „U” beams as in figure 4 and figure 5. As it can be seen, there are the same values for pulsations and frequencies like in table 3. It means that the movements inside the subsystems (X,φ_y) and (Y,φ_x) are very weakly coupled, almost uncoupled.

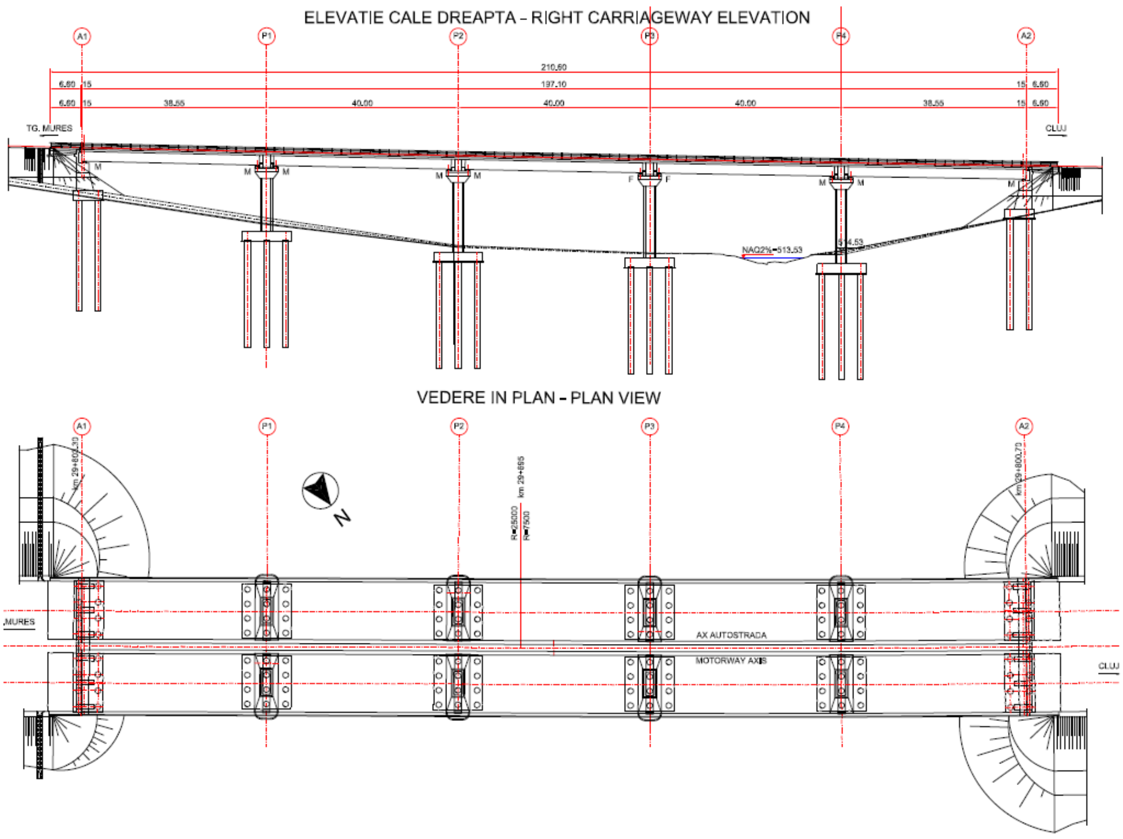

Figure 2 Elevation and plan view of the bridge (viaduct) on the Romanian motorway A3 at KM 29+602,75↔KM 29+801,25

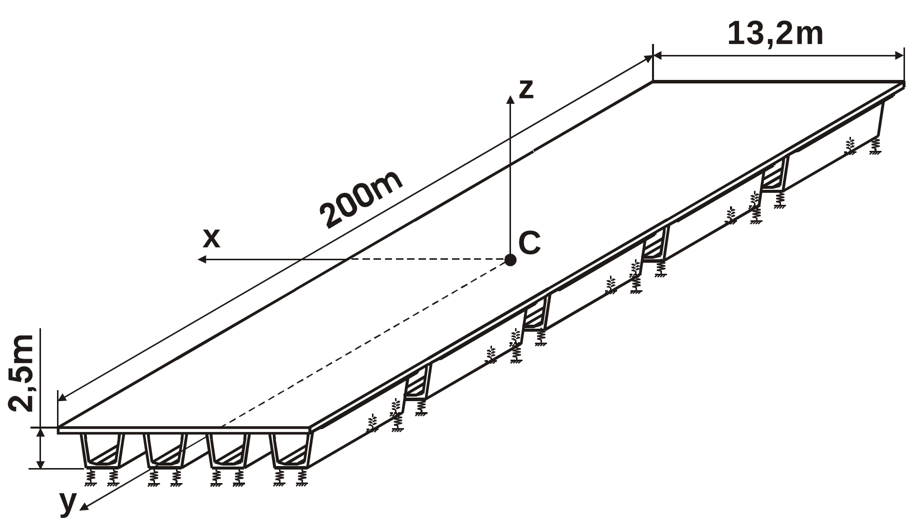

Figure 3 The model of the bridge beared on eighty neoprene supports

Table 1. Inertial characteristics (central and principal axis system)

Denomination		Unit	Arch of the viaduct (4 beams)	Viaduct (20 beams)
Mass m		kg	992,000	4,960,000
Products of inertia		$Kg\cdot m^2$	$J_{xy} = J_{yz} = J_{zx} = 0$	
Moments of inertia	J_x	$Kg\cdot m^2$	120.533×10^6	16.025×10^9
	J_y	$Kg\cdot m^2$	15.133×10^6	73.270×10^6
	J_z	$Kg\cdot m^2$	134.091×10^6	16.092×10^9

Table 2. Positions of the neoprene bearings

Bearing and coordinates [m]															
i	**x_i**	**y_i**	**z_i**	**i**	**x_i**	**y_i**	**z_i**	**i**	**x_i**	**y_i**	**z_i**	**i**	**x_i**	**y_i**	**z_i**
1	-5,5	-98,05	-1,45	**21**	1,1	-58,05	-1,45	**41**	-5,5	18,05	-1,45	**61**	1,1	58,05	-1,45
2	-4,4	-98,05	-1,45	**22**	2,2	-58,05	-1,45	**42**	-4,4	18,05	-1,45	**62**	2,2	58,05	-1,45
3	-2,2	-98,05	-1,45	**23**	4,4	-58,05	-1,45	**43**	-2,2	18,05	-1,45	**63**	4,4	58,05	-1,45
4	-1,1	-98,05	-1,45	**24**	5,5	-58,05	-1,45	**44**	-1,1	18,05	-1,45	**64**	5,5	58,05	-1,45
5	1,1	-98,05	-1,45	**25**	-5,5	-21,95	-1,45	**45**	1,1	18,05	-1,45	**65**	-5,5	61,95	-1,45
6	2,2	-98,05	-1,45	**26**	-4,4	-21,95	-1,45	**46**	2,2	18,05	-1,45	**66**	-4,4	61,95	-1,45
7	4,4	-98,05	-1,45	**27**	-2,2	-21,95	-1,45	**47**	4,4	18,05	-1,45	**67**	-2,2	61,95	-1,45
8	5,5	-98,05	-1,45	**28**	-1,1	-21,95	-1,45	**48**	5,5	18,05	-1,45	**68**	-1,1	61,95	-1,45
9	-5,5	-61,95	-1,45	**29**	1,1	-21,95	-1,45	**49**	-5,5	21,95	-1,45	**69**	1,1	61,95	-1,45
10	-4,4	-61,95	-1,45	**30**	2,2	-21,95	-1,45	**50**	-4,4	21,95	-1,45	**70**	2,2	61,95	-1,45
11	-2,2	-61,95	-1,45	**31**	4,4	-21,95	-1,45	**51**	-2,2	21,95	-1,45	**71**	4,4	61,95	-1,45
12	-1,1	-61,95	-1,45	**32**	5,5	-21,95	-1,45	**52**	-1,1	21,95	-1,45	**72**	5,5	61,95	-1,45
13	1,1	-61,95	-1,45	**33**	-5,5	-18,05	-1,45	**53**	1,1	21,95	-1,45	**73**	-5,5	98,05	-1,45
14	2,2	-61,95	-1,45	**34**	-4,4	-18,05	-1,45	**54**	2,2	21,95	-1,45	**74**	-4,4	98,05	-1,45
15	4,4	-61,95	-1,45	**35**	-2,2	-18,05	-1,45	**55**	4,4	21,95	-1,45	**75**	-2,2	98,05	-1,45
16	5,5	-61,95	-1,45	**36**	-1,1	-18,05	-1,45	**56**	5,5	21,95	-1,45	**76**	-1,1	98,05	-1,45
17	-5,5	-58,05	-1,45	**37**	1,1	-18,05	-1,45	**57**	-5,5	58,05	-1,45	**77**	1,1	98,05	-1,45
18	-4,4	-58,05	-1,45	**38**	2,2	-18,05	-1,45	**58**	-4,4	58,05	-1,45	**78**	2,2	98,05	-1,45
19	-2,2	-58,05	-1,45	**39**	4,4	-18,05	-1,45	**59**	-2,2	58,05	-1,45	**79**	4,4	98,05	-1,45
20	-1,1	-58,05	-1,45	**40**	5,5	18,05	-1,45	**60**	-1,1	58,05	-1,45	**80**	5,5	98,05	-1,45

Table 3. Natural pulsations and frequencies (on the six degrees of dynamic freedom)

System	Direction	X	Y	Z	φ_x	φ_y	φ_z
Arch of the viaduct (4 beams)	p [rad/s]	7.13	7.13	102.39	167.67	97.83	11.30
	f [Hz]	1.13	1.13	16.30	26.69	15.60	1.80
Viaduct (20 beams)	p [rad/s]	7.13	7.13	102.39	105.49	97.83	7.34
	f [Hz]	1.13	1.13	16.30	16.79	15.60	1.17

Table 4 Modal analysis for an arch (section) of the viaduct (decoupled subsystems)

Subsystem	Pulsations	Frequencies	Eigenvalues
(X,φ_y)	$p_1 = 7.13 rad/s$	$f_1 = 1.13 Hz$	$\mu_1 = 0.000509 rad/m$
	$p_2 = 97.83 rad/s$	$f_2 = 15.60 Hz$	$\mu_2 = -128.824 rad/m$
(Y,φ_x)	$p_3 = 7.13 rad/s$	$f_3 = 1.13 Hz$	$\mu_3 = -0.000002 rad/m$
	$p_4 = 167.67 rad/s$	$f_4 = 26.69 Hz$	$\mu_4 = 379.750 rad/m$
(Z)	$p_5 = p_Z = 102.39 rad/s$	$f_5 = f_Z = 16.30 Hz$	-
(φ_z)	$p_6 = p_{\varphi_z} = 11.30 rad/s$	$f_6 = f_{\varphi_z} = 1.80 Hz$	-

Table 5 Modal analyze for the viaduct (decoupled subsystems)

Subsystem	Pulsations	Frequencies	Eigenvalues
(X, φ_y)	$p_1 = 7.13 rad/s$	$f_1 = 1.13 Hz$	$\mu_1 = 0.000509 rad/m$
	$p_2 = 97.83 rad/s$	$f_2 = 15.57 Hz$	$\mu_2 = -128.824 rad/m$
(Y, φ_x)	$p_3 = 7.13 rad/s$	$f_3 = 1.13 Hz$	$\mu_3 = -0.000002 rad/m$
	$p_4 = 105.49 rad/s$	$f_4 = 16.79 Hz$	$\mu_4 = 149.916 rad/m$
(Z)	$p_5 = p_Z = 102.39 rad/s$	$f_5 = f_Z = 16.30 Hz$	-
(φ_z)	$p_6 = p_{\varphi_z} = 7.34 rad/s$	$f_6 = f_{\varphi_z} = 1.17 Hz$	-

The figures from table 5 show the values of the natural pulsations and frequencies and of the eigenvalues for the decoupled subsystems (with coupled movements) for the entire bridge composed from five sections (arches) considered being identical as in figure 3. As for the arches, the movements inside the subsystems with coupled movements (X, φ_y) and (Y, φ_x) of the viaduct are very weak coupled, almost uncoupled.

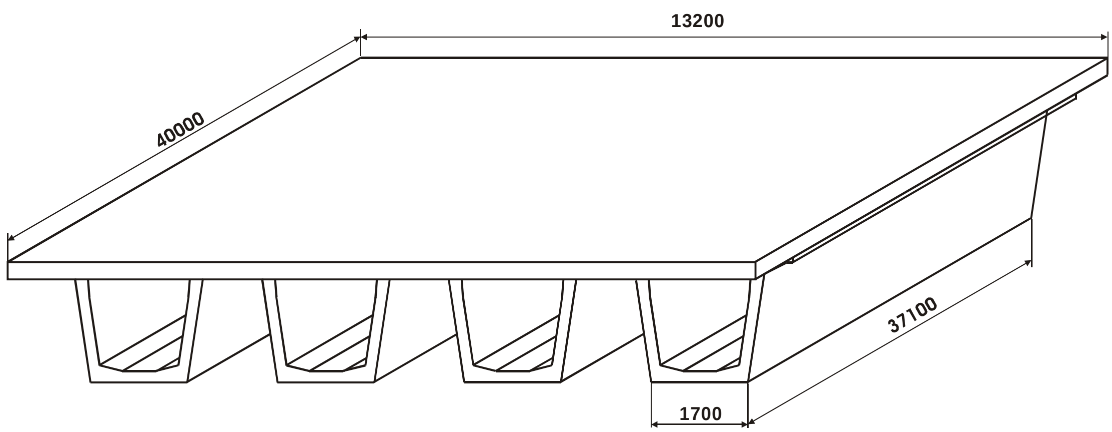

Figure 4 The model of an arch of the viaduct

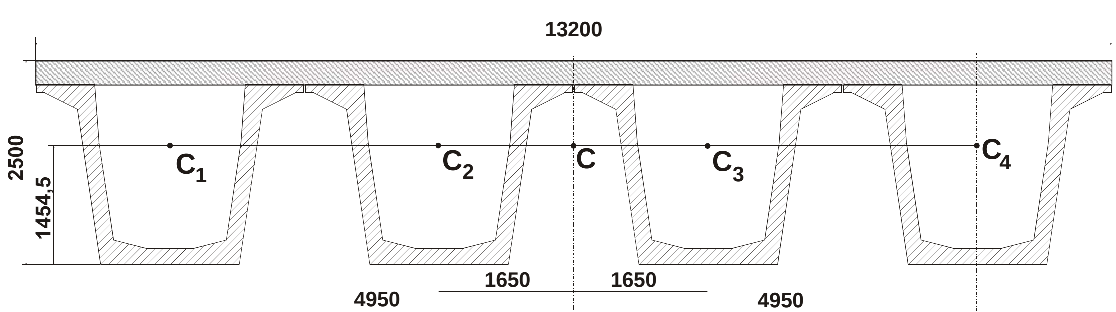

Figure 5 The model of an arch of the viaduct (transversal section)

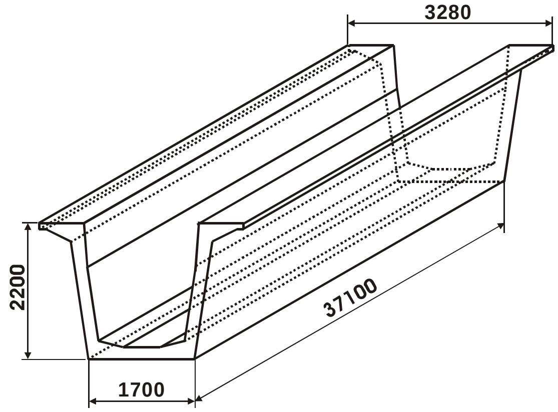

Figure 6 The reinforced concrete „U” beam (axonometric view)

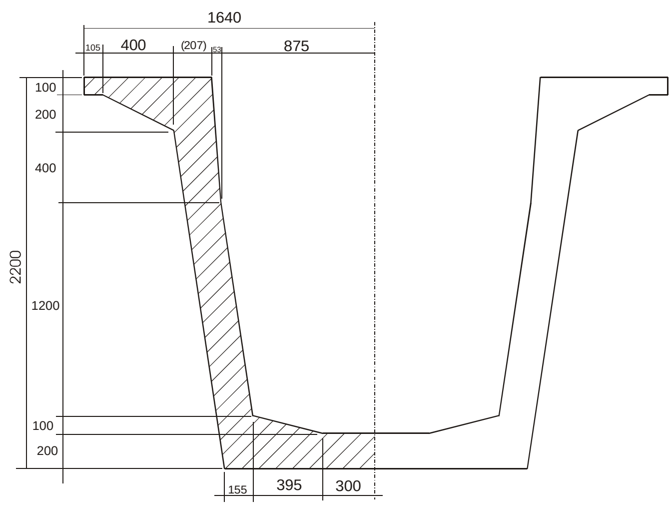

Figure 7 The reinforced concrete „U” beam (transversal section)

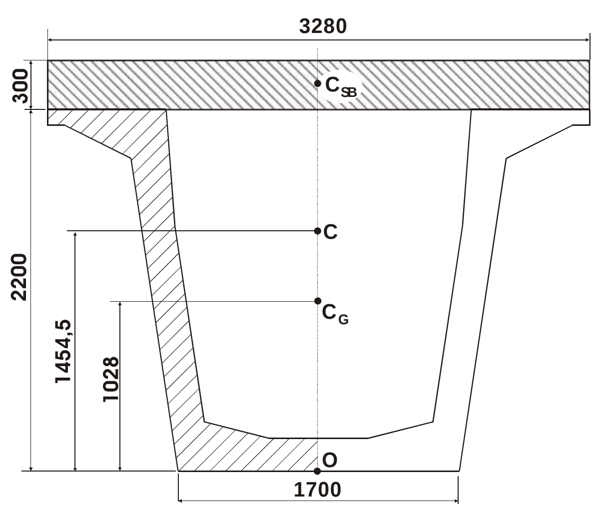

Figure 8 The „U” beam with concrete cover (transversal section)

4. CONCLUSIONS

a) modeling a rigid solid with elastic or viscous-elastic bearings and symmetries (structural, inertial, bearings) lead to more simple linear mathematical models, with differential equations decoupled into subsystems easier to solve; in this case, we can highlight the influences of different kinds of characteristics (dimensions, masses, inertia, stiffness) on the dynamic parameters of the rigid solid (natural pulsations/frequencies, eigenvalues);

b) if the physical model of the rigid solid permits to chose a Cartesian coordinate system which is central and principal, then the differential equations of motion are coupled only by the coefficients outside the principal diagonal of elasticity matrix (elastic coupling of movements), eventually by the dissipation coefficients from the viscous damping matrix if they are significant;

c) comparing the values of the pulsations/frequencies from tables 3, 4 and 5, we can say that the movements inside the subsystems are almost uncoupled on the "directions" ($X, Y, Z, \varphi_x, \varphi_y, \varphi_z$); also the values very small or very big of the eigenvalues can explain the quasidecoupling of the movements inside the subsystems;

d) analyzing the values from table 4 (for the arches), we can find a group of three natural frequencies in the domain 1.1÷1.2 Hz, another one in the domain 15.6÷16.3 Hz and the 6-th frequency being much bigger (26.69 Hz); this grouping of frequencies and the big differences between the values of domains' limits can be explained by the significant differences between the bearings stiffness on vertical axis Cz (compression effort) and on horizontal plane xCy (shear efforts);

e) analyzing the values from table 5 (for the entire bridge), we can find a group of three natural frequencies in the domain 1.1÷1.2 Hz and another three in the domain 15.6÷16.8 Hz; in this case of simulation, the pitch movement (φ_x) of the viaduct, which is almost decoupled from the forward-back movement (Y), has a natural frequency smaller than the pitch movement of a single arch because of a bigger value of the moment of inertia J_x mainly;

f) the mass characteristics and the moment of inertia for the entire bridge and for the arches were calculated on the basis of the sizes of "U" beam (according to fig. 6 and fig. 7) and those of the beam with the cover of concrete plate of 300mm thickness. (fig. 8)

REFERENCES

[1] **Bratu, P.**, *Vibraţiile sistemelor elastice*, Editura Tehnică, Bucureşti, 2000.

[2] **Bratu, P.**, *Izolarea şi amortizarea vibraţiilor la utilajele de construcţii*, Redacţia publicaţiilor pentru construcţii, Bucureşti, 1982.

[3] **Bratu, P.**, *Sisteme elastice de rezemare pentru maşini şi utilaje*, Editura Tehnică, Bucureşti, 1990.

[4] **Bratu, P., Drăgan, N.**, *L'analyse des mouvements désaccouplés appliquée au modèle de solide rigide aux liaisons élastiques*, Analele Universităţii "Dunărea de Jos" din Galaţi, Fascicula XIV, 1997.

[5] **Buzdugan, Gh., Fetcu, L., Radeş, M.**, *Vibraţii mecanice*, Ed. Didactică şi Pedagogică, Bucureşti, 1982.

[6] **Buzdugan, Gh.**, *Izolarea antivibratorie*, Ed. Academiei Române, Bucureşti, 1993.

[7] **Drăgan, N.**, *Contribuţii la analiza şi optimizarea procesului de transport prin vibraţii - teză de doctorat*, Universitatea "Dunărea de Jos", Galaţi, 2001.

[8] **Harris, C.M., Crede, C.E.**, *Şocuri şi vibraţii* vol. I-III, Ed. Tehnică, Bucureşti, 1967-1969.

[9] **Inman, D.**, *Vibration with Control*, John Wiley and Sons Ltd., New Jersey, 2006.

[10] **Rao, S.**, *Mechanical Vibrations Fourth Edition*, Pearson Education Inc., New Jersey, 2004.

[11] **Rădoi, M., Deciu, E.**, *Mecanica*, Editura Didactică şi Pedagogică, Bucureşti, 1977.