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ABOUT SIGNALS' SPECTRUM

Assoc. Prof. Dr. Eng. Luiza Grigorescu
Prof. Dr. Eng. Gheorghe Oproescu
Assoc. Prof. Dr. Eng. Ioana Diaconescu
"Dunarea de Jos" University of Galati, Romania

ABSTRACT

Due to this Fourier series exclusive interpretation applied to periodical signals, which shows that a stationary harmonic signal with a single pulsation ω_0 cannot contain components of other pulsations but its own pulsation ω_0, there is the opinion that stationary harmonic signals cannot stimulate physical systems which have their own pulsations different from ω_0. The authors' research have demonstrated that this theory is not true. More than this, signals considered identical, that is with identical amplitudes, phases and pulsations, are not identical from the spectral point of view if the signals have different durations and this can be noticed in many practical situations.

KEYWORDS: periodical signal, Fourier series, harmonics, amplitude, errors, spectral component, spectrum

1. INTRODUCTION

A signal is considered and interpreted as a variable that defines a time dependent physical phenomenon. As a generalized form, the signal is analytically defined, as a real function $f(t)$, with a single real variable that is t (=time). While with respect to commonly accepted criteria, the signals were classified into several categories [1], for this paper there were selected only the following categories:

Table 1 Signal categories

<table>
<tr><td colspan="5">Signal</td></tr>
<tr><td colspan="4">Deterministic</td><td>Random</td></tr>
<tr><td colspan="3">Periodical</td><td>Non periodical</td><td></td></tr>
<tr><td colspan="2">Harmonic</td><td colspan="2">Non-harmonic</td><td></td></tr>
<tr><td>Stationary</td><td>Non-Stationary</td><td>Stationary</td><td>Non-Stationary</td><td></td></tr>
</table>

The best known and also the most commonly used periodical signals will be shortly presented below:

The signal $f(t)=A(t)sin(\omega_0 t+\phi)$ is a harmonic signal, where $A(t)$ is the amplitude, ω_0 is its own pulsation, φ is initial phase; the signal is stationary for $A(t)=const$ and non-stationary for $A(t)\neq const$. If $A(t)=Ce^{-at}$, then the signal is called dampened signal, where *a* is the dampening factor [4].

The stationary non-harmonic signal can be found under many shapes, the most common of them being the "saw tooth" signal, originating from $f(t)=a(t-kT_0)$, the rectangular signal, originating from

$$f(t)=\begin{cases} a, & t\in[kT_0,\ kT_0+\mu\, T_0) \\ 0, & t\in[kT_0+\mu T_0,\ kT_0+T_0) \end{cases},$$

the "impulse train" signal that originates from

$f(t)=\begin{cases} a, & t=kT_0 \\ 0, & t\neq kT_0 \end{cases}$ where T_0 is its own period $T_0=2\pi/\omega_0$, $k=int(t/T_0)$, int is the "whole part of..." function, μ is a subunit and positive coefficient which characterizes the so-called filling factor of the signal, for $\mu=0,5$ the signal has thelevel a during the first half of the period and level 0 for the other second half. If the amplitude (the maximum value of $f(t)$ signal on a T *period*) is constant in time, it means that the signals are stationary, if the amplitude is variable in time, the signals are known as non-stationary.

Time t is only considered for positive values, $t\geq 0$.

When talking about deterministic and non-periodical signals (signals that follow a known and reproducible rule and contain no repetitive sequences within their entire duration), we refer to three categories of signals that are by now classics:

- step signal $f(t)=\begin{cases} a, & t\geq 0 \\ 0, & t<0 \end{cases}$,
- ramp signal $f(t)=\begin{cases} at, & t\geq 0 \\ 0, & t<0 \end{cases}$ and
- impulse signal (or Dirac function), which could also be written as $\delta(t)$ and be defined in many ways for different technical needs.

We will use the simplest definition of the Dirac function, characterized by two simultaneous valid conditions $\delta(t)=\begin{cases} +\infty & t=a \\ 0, & t\neq a \end{cases}$, $\int_{-\infty}^{+\infty}\delta(t)dt=1$. Besides the above signals there are infinitely more non periodical signals that can be artificially produced or that may be received from the environment, such as the sound signal from a concert, where the signal rule is provided by the musical score and each instrument's specific resonance.

Random signals are those signals that cannot be described by a rule. That is why random signals cannot be considered deterministic signals because there is no connection between cause (the rule) and effect (the signal) through within reproducible conditions. Such examples can be the evolution of the car's engine RPM during its running life, daily air temperature variation at a certain location and so on. Theoretically, some artificially produced functions can be interpolated with such random signals, given specific time limits $t\in[t_1,t_2]$ and the acceptance of a certain amount of inaccuracy, thus obtaining a deterministic signal. While this is the general approach for interpreting real life signals by deterministic signals, there are two issues that must be solved during this process:

-finding the best adequate interpolation functions

-eliminating the errors and interference during the signal's acquisition.

All the signals described above are considered continuous signals, although the mathematically speaking some of them, such as the saw tooth, rectangular or impulse signals offer obvious discontinuities. In spite of this, the mathematical operations of integration, derivation and operational computation are done by ignoring the inherent inaccuracy. For some applications, such as Fourier series coefficients determination, only Dirichlet conditions are enough (the function which describes the signal is bound, has a finite number of discontinuities and finite extremes during its period), conditions which cannot be fulfilled by the impulse signal only.

Mathematics creates the possibility that every stationary periodical non-harmonic signal with ω_0 pulsation be interpolated with an infinite series of stationary harmonic functions whose pulsation is a multiple of the signal pulsation $\omega=n\omega_0$, $n=1,2,3\cdots\infty$. Based on this mathematical artifice, we are talking about Fourier series, one can explain the fact that any periodical non-harmonic signal has within sources able to stimulate a wide range of physical systems with own frequency equaling any signal components $n\omega_0$, $n=1,2,3\cdots\infty$ although not necessarily with equal signal pulsations ω_0. Also, from Fourier series theory results that a pure harmonic signal doesn't contain signals with superior pulsations $n\omega_0$.

Due to this Fourier series exclusive interpretation applied to periodical signals, which shows that a stationary harmonic signal with a single pulsation ω_0 cannot contain components of other pulsations but its own pulsation ω_0, there is the opinion that stationary harmonic signals cannot stimulate physical systems which have their own pulsations different from ω_0. The authors' research demonstrated that this theory is not true [3], [5]. More than this, signals considered identical, that is with identical amplitudes, phases and pulsations, are not identical from the spectral point of view if the signals have different durations and this can be noticed in many practical situations.

2. FOURIER SERIES AND ITS PROPERTIES

A stationary periodical signal $g(t)$ with the period $T_0 = \frac{2 \cdot \pi}{\omega_0}$, where ω_0 is the signal pulsation, which fulfils the Dirichlet conditions, can be represented by a mathematical series whose terms are harmonic functions with pulsations multiple of the ω_0 pulsation. The ω_0 pulsation is called fundamental pulsation and the harmonic function with the pulsation equal to ω_0 is called fundamental harmonic. Harmonic functions with pulsations $n \cdot \omega_0$, $n = 2, 3, \cdots$ are called n order harmonics. The general form for the series of harmonic functions is:

$$f(t) = \sum_{n=0}^{\infty} \left[a_n \cdot cos(n \cdot \omega_0 \cdot t) + b_n \cdot sin(n \cdot \omega_0 \cdot t) \right] \quad (1)$$

where the series has an infinite number of members and their values are:

$$a_0 = \frac{1}{T} \int_0^{T_0} g(t) \cdot dt; \quad b_0 = 0;$$

$$a_n = \frac{2}{T} \int_0^{T_0} g(t) \cdot cos(n \cdot \omega_0 \cdot t) \cdot dt; \quad (2)$$

$$b_n = \frac{2}{T} \int_0^{T_0} g(t) \cdot sin(n \cdot \omega_0 \cdot t) \cdot dt; \quad n = 1, 2, \cdots$$

The constant a_0, which is a measure of signal $g(t)$ asymmetry with respect to the abscissa. Each harmonic component contains two terms from the same pulsation creating a trigonometric equivalent function:

$$f(t) = a_0 + \sum_{1}^{\infty} A_n \cdot sin(n \cdot \omega_0 \cdot t + \phi_n) \quad (3)$$

where the amplitude A_n and phase φ_n result from the coefficients identification:

$$\begin{cases} A_n = \sqrt{a_n^2 + b_n^2} \\ \phi_n = arctg \dfrac{a_n}{b_n} \end{cases} \quad (4)$$

By using the (3) formula, with coefficients calculated according to the relation (4), one finds that a specific order n harmonic is a stationary sinusoidal signal with its own amplitude and phase. In expressions (3) and (4) one notices that a certain harmonic of order n is actually a signal produced by a rotating vector $\vec{v}_n$ (complex number), having modulus A_n and phase ϕ_n.

$$\vec{v}_n = A_n \cdot e^{j \cdot (n\omega_0 t + \phi_n)} \quad (5)$$

where components b_n are on the real axis and a_n are on the imaginary axis.

To determine the harmonics of a periodic signal is not only a theoretical problem which allows decomposing a periodic function into other periodic functions. Harmonics existence are strongly felt in practice because a non harmonic periodic signal generates an infinite number of excitation sources having frequencies equal to multiples of the basic signal frequency and these sources produce obvious effects by stimulating the physical systems with pulsations (frequencies) equal to any multiple of the signal's own pulsation (frequency) [1] (pp.127). Frequency multipliers used in radiotechnics are based solely on this obviously very real phenomenon. For multiplication, a stationary harmonic signal is distorted to generate harmonic components with pulsations $n\omega_0$ whence, by filtration, the component with the value n that is desired is extracted, n usuallly being equal to 2,...5.

This phenomenon is also found when referring to the mechanical systems, respectively a non-harmonic signal generating sources of excitation which get in resonance with components of the system. A good example is given by the non-harmonic signals such as earthquakes which produce damages to constructions or parts of constructions that have their own pulsation equal to harmonics of the earthquake.

A non-harmonic periodic signal is the better defined in Fourier series, the more components of the series are identified, respectively the higher n gets. In reality, one also gets a limitation here: in calculating the coefficients of the Fourier series, and even in the series itself, one uses the harmonic functions $\sin(n\omega_0 t), \cos(n\omega_0 t)$ for which, the higher the value of n gets, the higher is the value of the parameter one needs to evaluate, while in mathematics, it is well known that expressions $sin(\infty)$, $cos(\infty)$ are indeterminate.

By using a computer to calculate the series elements the non-determination situation described above is rapidly reached due to the way in which numbers are represented by the operating system or programming language, since numbers are only represented with a finite number of figures. The below examples clearly show this fact.

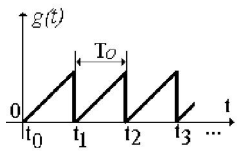


Fig. 1 - Tooth of the saw signal

Be a saw tooth signal which is analytically defined in the above, having the period $T_0 = 1$ second and its amplitude=*1,* fig. 1. In fig. 2, in the upper side we show the shape of the first eight harmonics and in the lower side the initial signal, reconstructed from ten harmonics.

The value of the constant component a_0 is 0,99401 and the amplitude of the tenth order harmonic A_{10} is 0,01593.

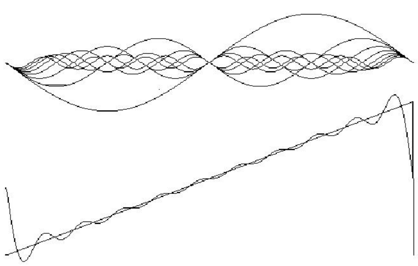
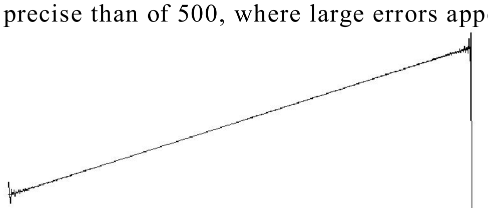


Fig. 2 - Reconstruction of ten harmonics

By increasing the number of harmonics one would expect the signal reconstruction to be more accurate, maybe with the exception of interval limits, where the mathematical discontinuity is also more pronounced. By increasing the number of harmonics, like in fig. 3 and 4, one notices a contradictory situation: reconstruction out of 300 harmonics is more precise than of 500, where large errors appear.



a_0=0,99401; A_{300}=0,00105

Fig. 3 - Recontruction out of 300 harmonics

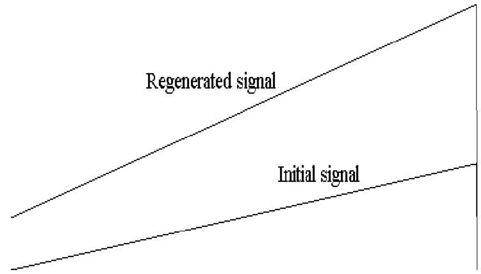


Fig. 4 - Reconstruction out of 500 harmonics

To find the source of this deviation one has to research the evolution of the harmonics amplitude in the studied cases, fig. 5.

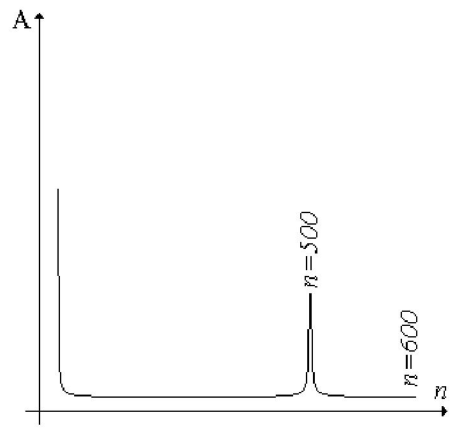


Fig. 5 - The amplitude of the first 600 harmonics

If for a number up to 300-400 harmonics their amplitude continually decreases at the same time with the increase of the given harmonics number of order, by continuing to increase at number of harmonics, their amplitude begins to increase again up to the value $A_{500} = 0,497$. Continuing to increase the number of harmonics, their amplitude decreases again followed once more by an increase for order high value. This variation of the amplitude by the increase of the number of order, is not due to the structure of known relations of calculation but actually to the computer having to operate with hard conditioned relations. The cause of the deviations from figure 4, where the signal is reconstructed out of 500 harmonics, can be thus found in the particular way in which the digital computers operate with numbers which have a finite number of figures and, due to the truncation errors that appear when a large number of computations is being done, one may end up experiencing big errors.

Subsequently, one can say of the Fourier series that:

- It is a series made of harmonic functions with pulsations equal to multiples of the pulsation of the non-harmonic periodic signal whence it originates;
- The harmonic functions which compose the Fourier series are real sources of excitation for the physical systems which have their own pulsations equal to one of the harmonic pulsation;
- Each harmonics amplitude has a finite value, usually considered as continuously decreasing by the order of the harmonic, although there may occur situations where the decrease is not continued;
- The harmonic functions come together in a discrete spectrum of pulsations contained in the base signal;
- The characteristics of each spectrum's harmonic, meaning amplitude and phase, are independent of the signal duration;
- The Fourier series doesn't show if, among the discrete harmonic components, it has others able to excite various other oscillating systems.

3. COMPARISON WITH THE FOURIER TRANSFORM

Let's now compare the discrete spectral components given by the harmonics from the Fourier series with the continuous spectral components provided by the Fourier transform. For a signal described by the real function $g(t)$, the Fourier transform leads to the function $G(j\omega)$ [4](pp. 14):

$$G(j\omega)=\int_{-\infty}^{+\infty} e^{-j\omega t} g(t)\,dt \tag{6}$$

with $j=\sqrt{-1}$ which, after being submitted to the below transformation:

$$\begin{aligned}G(j\omega)&=\int_{-\infty}^{+\infty} cos(\omega t) g(t)\,dt-\\&-j\int_{-\infty}^{+\infty} sin(\omega t) g(t)\,dt=\\&=Re(\omega)+j\,Im(\omega)\end{aligned} \tag{7}$$

becomes a complex function with a real part $Re(\omega)$ and an imaginary one $Im(\omega)$. The modulus of this function $S(\omega)=|G(j\omega)|$ is calculated by:

$$S(\omega)=\sqrt{Re^2(\omega)+Im^2(\omega)} \tag{8}$$

and one obtains a function dependent on the pulsation (or frequency), which is either named frequency characteristic should it refer to the behavior of one element of the signal's propagation chain or spectral function should it refer to the spectral components of a signal $g(t)$. For any time function $g(t)$, the values from (6)---(8) are to be found easily on the numerical way [2](pp. 123-124). Let's consider the pure harmonic signal:

$$g(t)=sin(\omega_o t) \tag{9}$$

which is fully known for $t\in[0\cdots 2\pi/\omega_o]$. For the spectrum determined via the Fourier transform using (8), the integration will be done between variable limits, those being *0* and $2n\pi/\omega_o$ (*n* is a multiple of the period $2\pi/\omega_0$), in order to detect a possible influence of the signal's duration upon its spectrum, duration which doesn't appear in the Fourier series. The form below of the spectral function is obtained on the analytical calculation:

$$S(\omega)=2\omega_0\left|\frac{sin\left(n\pi\frac{\omega}{\omega_o}\right)}{\omega^2-\omega_o^2}\right| \tag{10}$$

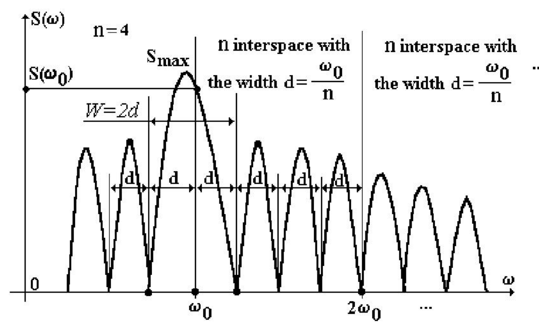


Fig.6 - Continuous spectrum of the harmonical pure signal

Figure 6 shows the (10) function for *n=4*. Analyzing expression (10) one notices some interesting aspects:

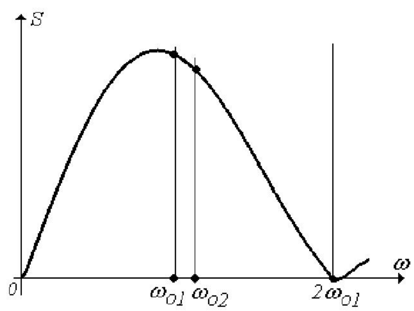
- In the point determined by $\omega=\omega_0$, on the abscissa, the function has the value:

$$S(\omega_0)=2\omega_0 \lim_{\omega\to\omega_o}\left|\frac{sin(n\pi\omega/\omega_o)}{\omega^2-\omega_o^2}\right|=n\frac{\pi}{\omega_o} \tag{11}$$

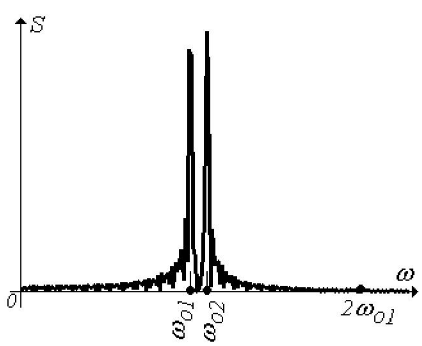
- The peak of the spectral function, meaning the maximum amplitude, is not obtained for the pulsation $\omega = \omega_0$.
- Amplitude S for $\omega = \omega_0$ is increasing by the signal duration, so that for $n \to \infty$, $S \to \infty$.
- Signal (9) has in fact infinite pulsations should it have a finite duration in time.

The above shows that even a pure harmonic signal, which ought to have a frequency spectrum with a single pulsation, in reality has a very large spectrum, with many pulsations, the spectrum becoming narrow only for a single value $\omega = \omega_0$ and only if the signal is infinitely lasting, $n \to \infty$. Since the signals have, in practice, a finite duration and are thus lasting for only several periods such as in the case of earthquakes or mechanical shocks, it means that their spectrum is actually very rich and they can excite an infinite number of devices which could have their own pulsation equal to some pulsation generated by the signal in its spectrum. Having a wide spectrum for short durations for a signal has one more inconvenient. Let's consider a signal composed of two stationary harmonic components:

$$g(t) = A_1 \sin(\omega_{01} t + \phi_1) + A_2 \sin(\omega_{02} t + \phi_2) \qquad (12)$$



n=1



n=50

Fig. 7 - The spectrum of a signal composed of two harmonic signals with near pulsations

where we can use for example: $A_1 = 1,\ A_2 = 1,\ \omega_{01} = 100\pi,\ \omega_{02} = 110\pi,$ $\phi_1 = \phi_2 = 0$ and which is lasting initially for only one period, and then for 50 periods with the value of $2\pi/\omega_{01}$.

The spectrum from figure 7 shows that for the short length signal the spectrum's width is so big that it doesn't allow anymore the identification of its components' pulsations, the bandwidth exceeds the difference between the two pulsations ω_{01}, ω_{02}. If the signal is lasting more, for example 50 periods, the spectral bandwidths get narrow and they show very clearly the existence of the two spectral components.

4. CONCLUSIONS

The elements of the Fourier series represent the spectral components of a given signal, only if the signal is lasting for very long time. For signals with a short duration of time the larger the spectrum the shorter the duration is. This explains the multitude of systems with various own pulsations excited for very short periods by short duration signals such as earthquakes (mechanical systems) or radio electrical interferences produced by the engines ignition. That is why earthquakes destroy walls, pillars, consoles or chimneys with various dimensions and shapes and radio electrical interferences are detected up to frequencies of hundreds MHz.

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