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ABOUT NON-AUTONOMOUS VIBRATIONS OF MECHANICAL SYSTEMS

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ABSTRACT

Non-linear vibrations of mechanical systems are described by differential equations of the second order, which are generally written $\ddot{x}+f(x,\dot{x},t)=0$. *In this work we study non-autonomous vibrations of mechanical systems with mathematical models* $\ddot{x}+f(x,\dot{x},t)=0$ *for which we determine exact analytical solutions converting them into Riccati special equations and then using substitutions of functions in differential equations with separable variables or linear equations with variable coefficients.*

KEYWORDS: mechanical, system, vibrations, equation, non-linear.

1. Introduction

In a series of mechanical phenomena there are non-linear vibrations and their mathematical shaping leads to differential equations or systems of non-linear differential equations. We study phenomena which are described by differential equations as:

$$\ddot{x}+f(x,\dot{x},t)=0 \qquad (1)$$

which come from Newton equation which equalizes the elastic forces with movements.

Particularly, we consider autonomous vibrations of mechanical systems, which are characterized, as [1], of non-linear differential equations:

$$\ddot{x}+f(x,\dot{x})=0, \qquad (2)$$

or non-autonomous vibrations described by non-linear differential equation:

$$\ddot{x}+f(\dot{x},t)=0. \qquad (3)$$

In this work we demonstrate that for functions such as:

$$f\left(\dot{x},t\right)=a\dot{x}^2+\frac{1}{t^2}\left(bt^{\alpha}+c\right);$$

$$a,b,c,\alpha\in R \qquad (4)$$

we can determine the analytical exact solutions for non-linear differential equations (3) which describe the non-autonomous vibrations of a mechanical system.

2. Analytical solution

Employing (6) the differential equation (3) becomes:

$$\ddot{x}+a\dot{x}^2+\frac{1}{t^2}\left(bt^{\alpha}+c\right)=0 \qquad (5)$$

and substituting:

$$\dot{x}=y\left(t\right) \qquad (6)$$

we write:

$$\dot{y}+ay^2+\frac{1}{t^2}\left(bt^{\alpha}+c\right)=0, \qquad (7)$$

a Riccati special differential equation.

We shall perform a change of function

$$u(t) = t \cdot y(t) + k;\ k \in R \qquad (8)$$

and noticing that:

$$y(t) = \frac{1}{t}\left[u(t) - k\right]; \qquad (9)$$

$$\dot{y}(t) = \frac{1}{t} \cdot u'(t) - \frac{1}{t^2}\left[u(t) - k\right], \qquad (10)$$

the differential equation (7) becomes:

$$t \cdot u' + au^2 - a_1 u + b \cdot t^{\alpha} = 0, \qquad (11)$$

in which:

$$a_1 = 1 + 2ak \qquad (12)$$

and

$$ak^2 + k + c = 0. \qquad (13)$$

In order to continue solving the differential equation (7) it is compulsory that the algebraic equation (13) of second order with the k variable had real solutions, so it is checked up if

$$1 - 4ac \geq 0. \qquad (14)$$

In the differential equation (11) we introduce the substitutions

$$v = t^{a_1};\ a_1 \in R;\ a_1 \neq 0 \qquad (15)$$

and

$$u = v \cdot \beta(v) \qquad (16)$$

the equation becoming after the necessary derivations:

$$\beta' + \frac{a}{a_1}\beta^2 = -\frac{b}{a_1} \cdot v^{a_2} \qquad (17)$$

where:

$$a_2 = \frac{\alpha}{a_1} - 2. \qquad (18)$$

By changing the function:

$$z(v) = \frac{a}{a_1}\beta(v) \qquad (19)$$

we obtain from (17) the differential equation:

$$z' + z^2 = a_3 \cdot v^{a_2}, \qquad (20)$$

where:

$$a_3 = -\frac{ab}{a_1^2}. \qquad (21)$$

Before considering the general case, in order to determine the analytical solution, we discuss the particular cases for the differential equation (20). If $a_2 = 0$ the differential equation (20) has separable variables. If

$$a_2 = -2 \qquad (22)$$

the differential equation (20) is a Riccati equation and by changing the function:

$$z(v) = v \cdot w(v) \qquad (23)$$

we obtain :

$$v \cdot w' + w^2 - w - a_3 = 0 \qquad (24)$$

whence there results, after solving it as a equation with separable variables and after simple calculations :

$$\ln|v| = -\int \frac{dw}{w^2 - w - a_3} + c_1;\ c_1 \in R. \qquad (25)$$

In the general case we perform the change of function:

$$z(v) = v^{p+1} \cdot \frac{w'}{w};\ w \neq 0, \qquad (26)$$

where:

$$p = -\frac{1}{a_2 + 2}. \qquad (27)$$

and we obtain the liniar differential equation of the second order:

$$v\ w'' + p \cdot w' - w = 0. \qquad (28)$$

In all cases discussed, after finding the function $w = w(v)$ with (26), (23), (19), (16), (8) and (6) there is the analytical exact solution $x(t)$ of the differential equation (5).

We ask ourselves if the differential equation (28) can be solved.

It is generally as:

$$\ddot{w} + p_1(v)\,\dot{w} + p_2(v) \cdot w = 0. \qquad (29)$$

So as to solve it, we substitute:

$$w = w_1 \cdot y, \qquad (30)$$

where $w_1 = w_1(v)$ is a particular function which is to be determined, and y is a new unknown.

It results :

$$\dot{w} = w'_1 \cdot y + w_1 \cdot y' \; ; \quad (31)$$

$$\ddot{w} = w''_1 \cdot y + 2\, w'_1 \cdot y' + w_1 \cdot y'' \quad (31')$$

and replacing in the equation (29), then :

$$w''_1 y + w'_1 y' + w'_1 y' + w_1 y' + \\ + p_1(v)\,[w'_1 y + + w_1 y'] + \\ + p_2(x) \cdot w_1 y = 0 \quad (32)$$

or:

$$w_1 \cdot y'' + \left[2w'_1 + p_1(v) \cdot w_1\right] y' + \\ + \left[w''_1 + p_1(v) \cdot w'_1 + p_2(v)\; w_1\right] = 0 \quad (33)$$

We distinguish two situations:

I. If :

$$2w'_1 + p_1(v) \cdot w_1 = 0 \quad (34)$$

then:

$$\frac{w'_1}{w_1} = -\frac{p_1(v)}{2};$$

$$\ln w_1 = -\int \frac{p_1(v)}{2} dv + \ln c;\; c \in R,\; c > 0;$$

$$w_1 = c \cdot e^{-\frac{1}{2}\int p_1(v) dv}. \quad (35)$$

We shall consider, without restricting the generality, that c=1, and it results:

$$w_1 = e^{-\frac{1}{2}\int p_1(v) dv}, \quad (36)$$

and from (34):

$$w'_1 = -\frac{1}{2} p_1(v) \cdot w_1; \quad (37)$$

$$w''_1 = -\frac{1}{2}\left(\frac{dp_1(v)}{dv} \cdot w_1 + p_1(v)\; w'_1\right) = \\ = -\frac{1}{2}\left(\frac{dp_1(v)}{dv} \cdot w_1 - \frac{1}{2} p_1^2(v) \cdot w_1\right) = \\ = -\frac{1}{2}\left[\frac{dp_1(v)}{dv} - \frac{1}{2} p_1^2(v)\right] w_1. \quad (38)$$

Then:

$$w_1'' + p_1(v) \cdot w_1' + p_2(v) \cdot w_1 = \\ = \left[\frac{1}{4} p_1^2(v) - \frac{1}{2} \cdot \frac{dp_1(v)}{dv} - \right. \\ \left. - \frac{1}{2} p_1^2(v) + p_2(v)\right] \cdot w_1 = \\ = \left[p_2(v) - \frac{1}{4} p_1^2(v) - \frac{1}{2}\frac{dp_1(v)}{dv}\right] w_1. \quad (39)$$

and:

$$I(v) = p_2(v) - \frac{1}{4} p_1^2(v) - \frac{1}{2}\frac{dp_1(v)}{dv} \quad (40)$$

the equation (29) becomes :

$$y'' + I(v) \cdot y = 0 \quad (41)$$

which can be easily solved in some cases(for example $I(\nu)$=const. or multiple of $\frac{1}{\nu^2}$) or generally using the substitution $\nu = e^s$ it becoming a liniar non-homogeneus differential equation with constant coefficients.

II. Coming back to (33), if

$$w''_1 + p_1(v) \cdot w'_1 + p_2(v) \cdot w_1 = 0 \quad (42)$$

then w_1 is a particular solution of the equation (29) which becomes

$$w_1\, y'' + \left[2w'_1 + p_1(v)\, w_1\right] \cdot y' = 0 \quad (43)$$

and this makes possible for a one unit order reduction in with the substitution

$$y' = z,\;\; z = z(t) \quad (44)$$

and then we solve it as an equation with separable variables.

3. Example

The non-autonomus vibrations of a mechanical system are described by the non-linear differential equation.

$$\ddot{x} - \dot{x}^2 + \frac{2}{t^2} = 0, \quad (45)$$

so comparing it with (5) we have:

$$a=-1,\ \ b=2,\ \ c=0,\ \ \alpha=0. \qquad (46)$$

From (13) it results that $k=0$ or $k=1$.

We shall focus on the case when $k=0$, the other will be solved the same way and we shall harp back to it. From (12) we obtain $a_1=1$, and from (18) it results $a_2=-2$, and from (21) we know $a_3=2$, so the differential equation (24) will be written:

$$v\cdot w''+w^2-w-2=0. \qquad (47)$$

The equality (25) leads to :

$$\ln|v|=-\int\frac{dw}{w^2-w-2}+\ln c_1;$$
$$c_1\in R,\ c_1>0. \qquad (48)$$

We consider the constant of integration as a logarithm in order to simplify calculations.

From integral (48) it results:

$$\ln|v|=\ln c_1\left|\frac{w+1}{w-2}\right|^{1/3} \qquad (49)$$

and then by abdicating to the module and performing the calculations:

$$w(v)=2+\frac{3c_1^3}{v^3-c_1^3}. \qquad (50)$$

From substitution (23) we obtain:

$$z(v)=2v+\frac{3c_1^3 v}{v^3-c_1^3}, \qquad (51)$$

and from (19), (16), (15) şi (9) there results:

$$y(t)=-2t-\frac{3c_1^3 t}{t^3-c_1^3}. \qquad (52)$$

Using the substitution (6) we find the analytical exact solution of the differential equation (45)

$$x(t)=-t^2-3c_1^3\int\frac{tdt}{t^3-c_1^3}=$$

$$=-t^2-3c_1^2\ln\frac{(t-c_1)^{1/3}}{\left(t^2+c_1t+c_1^2\right)^{1/6}}-$$
$$-\sqrt{3}c_1^2\operatorname{arctg}\frac{2t+c_1}{c_1\sqrt{3}}+c_2, \qquad (53)$$

the constants c_1, c_2 will be obtained from the imposal of the initial conditions on the movement of the system.

Back to case k=1, the calculations lead to

$$a_1=-1;\ \ a_2=-2;\ \ a_3=2, \qquad (54)$$

so the expression of z(v) remains unchanged but from the substitutions (16) it results:

$$v=t^{-1}, \qquad (55)$$

and from (20), (17), (11) we obtain:

$$y(t)=\frac{1}{t^2}\left(1+\frac{3c_1^3t^3}{1+c_1^3t^3}\right). \qquad (56)$$

From the substitution (8), it will result the analytical exact solution:

$$x(t)=-\frac{1}{t}-\ln\frac{|1-c_1t|^{\frac{1}{c_1}}}{\left(1+c_1t+c_1^2t^2\right)^{\frac{c_1}{2}}}+$$
$$+\frac{c_1}{3\sqrt{3}}(2c_1-3)\operatorname{arctg}\frac{1+2c_1t}{\sqrt{3}}+c_2, \qquad (57)$$

the constants c_1, c_2 are calculated as in the previous case.

4. Conclusion

For the non-linear differential equations as (3) where functions $f\left(\dot{x},t\right)$ have the expression (6) and that characterizes the non-autonomous vibrations of the mechanical systems, we can determine analytical exact solutions which permit the analysis of movement.

References

[1] **P.P. Teodorescu** – *Sisteme mecanice. Modele clasice*, vol IV, Ed. Tehnică, Bucureşti,2002.