THE ANNALS OF "DUNAREA DE JOS" UNIVERSITY OF GALATI
FASCICLE XIV MECHANICAL ENGINEERING, ISSN 1224-5615
2010

THE ANALYTICAL EXACT SOLUTIONS OF NON-LINEAR DIFFERENTIAL EQUATIONS WHICH DESCRIBE THE PARAMETRIC FREE VIBRATIONS OF MECHANICAL SYSTEMS

Assoc.Prof.Math.Dr.Eng. Gh. Cautes
"Dunarea de Jos" University of Galati

ABSTRACT

The movements of many material systems can be described by differential equations that have coefficients that depend on time. It is difficult to determine the solutions of these equations. Although many concrete problems lead to non-linear differential equations where we can also find the term of variable damping, the classic equations that have been studied more were Hill or Mathieu, with periodic coefficient, that do not contain derivations of the first order. At this kind of equations, we reduce it to second order using substitutions as we will demonstrate.

In this work, we show that we can obtain analytical exact solutions for non-linear homogeneous differential equations with variable coefficient which describe the parametric free vibrations of mechanical systems.

KEYWORDS: vibration, mechanical system, non-linear

1. Introduction

Some mechanical systems have variable parameters in time such as the mass, the dimensions, the elastic coefficient or the damping coefficient. The mathematical pendulum with the variable length is that kind of a system.The oscillations of a system with variable parameters are described by homogeneous differential equation

$$\ddot{x} + p_1(t)\cdot\dot{x} + p_2(t)\cdot x = 0 . \qquad (1)$$

In [1] for the particular differential equation:

$$\ddot{x} + p(t)\cdot x = c\cdot x^{-\alpha} \qquad (2)$$

where $c, \alpha \in R$, there resulted the analytical exact solutions.

In [2] we have shown that we can find analytical approximate solutions for non-linear differential equations as

$$m\ddot{x} - ct^k x = 0 ;\ c, k \in R \qquad (3)$$

which describe the parametrical free vibrations of the mechanical systems that do not imply damping forces.

The theoretical concept used to determine solutions utilises the theory of Bessel functions.

2. Theoretical concept

We shall demonstrate that for equation (1) we can find analytical exact solutions in general situations. Afterwards, we shall exemplify with differential equations that describe parametric free vibrations of mechanical systems with variable parameters.

Going back to the differential equation with variable coefficients in time (1), according to [3], we substitute

$$x = x_1 \cdot y , \qquad (4)$$

where

$$x_1 = x_1(t) \qquad (5)$$

it is a particular function which will be determined while y is the unknown function.

Then

$$\dot{x} = x_1' \cdot y + x_1 \cdot y' \; ; \qquad (6)$$

$$\ddot{x} = x_1'' y + 2 \cdot x_1' \cdot y' + x_1 \cdot y'' , \qquad (7)$$

and after replacing in (1)

$$x_1'' y + x_1' y' + x_1' y' + x_1 y'' + + p_1(t)\left[x_1' y + x_1 y'\right] + p_2(t) x_1 y = 0, \quad (8)$$

or

$$x_1 y'' + \left[2x_1' + p_1(t) x_1\right] y' + + \left[x_1'' + p_1(t) x_1' + p_2(t) x_1\right] y = 0 . \quad (9)$$

There are two situations:
I. If

$$2x_1' + p_1(t) \cdot x_1 = 0 \qquad (10)$$

after considering a differential equation with separable variables we obtain

$$\frac{x_1'}{x_1} = -\frac{p_1(t)}{2} ; \qquad (11)$$

$$\ln x_1 = -\int \frac{p_1(t)}{2} dt + \ln c \; ; c \in R, c > 0; \quad (12)$$

$$x_1 = c \cdot e^{-\frac{1}{2}\int p_1(t) dt} . \qquad (13)$$

It is thought that c=1, without restricting the generality and it results

$$x_1 = e^{-\frac{1}{2}\int p_1(t) dt} , \qquad (14)$$

and from (10) , we write

$$x_1' = -\frac{1}{2} p_1(t) \cdot x_1 ; \qquad (15)$$

$$x_1'' = -\frac{1}{2}\left(\frac{dp_1(t)}{dt} \cdot x_1 + p_1(t) \cdot x_1'\right) = = -\frac{1}{2}\left(\frac{dp_1(t)}{dt} x_1 - \frac{1}{2} p_1^2(t) x_1\right) = = -\frac{1}{2}\left[\frac{dp_1(t)}{dt} - \frac{1}{2} p_1^2(t)\right] x_1 . \quad (16)$$

Then

$$x_1'' + p_1(t) x_1' + p_2(t) x_1 = = \left[\frac{1}{4} p_1^2(t) - \frac{1}{2}\frac{dp_1(t)}{dt} - -\frac{1}{2} p_1^2(t) + p_2(t)\right] \cdot x_1 = = \left[p_2(t) - \frac{1}{4} p_1^2(t) - \frac{1}{2} \cdot \frac{dp_1(t)}{dt}\right] \cdot x_1 . \quad (17)$$

If

$$I(t) = p_2(t) - \frac{1}{4} p_1^2(t) - \frac{1}{2}\frac{dp_1(t)}{dt} \qquad (18)$$

the equation (1) becomes

$$y'' + I(t) \cdot y = 0 \qquad (19)$$

that can be easily solved in some particular cases (for example I(t)=const. or multiple of $\frac{1}{t^2}$), or generally using the substitution

$$t = e^s \qquad (20)$$

it becoming a differential equation with constant coefficients.

3. Examples

We exemplify now for elastic systems whose parametric oscillations are characterized by liniar equations that verify the condition (10).

1. The parametric vibrations of a mechanical system are described by the non-linear homogenous differential equation

$$\ddot{x} + 2 \cdot \dot{x} + \left(1 - \frac{2}{t^2}\right) \cdot x = 0 \qquad (21)$$

and we shall obtain analytical exact solutions.

From the substitution (14), after calculating, we find

$$x_1 = e^{-t} , \qquad (22)$$

and from (18) it results

$$I(t) = -\frac{2}{t^2} . \qquad (23)$$

The considered equation becomes

$$y'' - \frac{2}{t^2} \cdot y = 0 \tag{24}$$

and with the substitution

$$t = e^s, \quad s = s(t) \tag{25}$$

now becomes

$$\frac{d^2 y}{ds^2} - \frac{dy}{ds} - 2y = 0, \tag{26}$$

when

$$\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = \frac{1}{t} \cdot \frac{dy}{ds},$$
$$\frac{d^2 y}{dt^2} = \frac{d^2 y}{ds^2} \cdot \frac{ds}{dt} \cdot \frac{1}{t} - \frac{dy}{dt} \cdot \frac{1}{t^2} =$$
$$= \frac{1}{t^2}\left(\frac{d^2 y}{ds^2} - \frac{dy}{ds}\right). \tag{27}$$

The liniar differential equation (26) with constant coefficients has a general solution

$$y = c_1 \cdot e^{-s} + c_2 \cdot e^{2s}; \quad c_1, c_2 \in R \tag{28}$$

and then

$$x(t) = e^{-t} \cdot \left(c_1 \cdot e^{-\ln t} + c_2 \cdot e^{2\ln t}\right) =$$
$$= e^{-t}\left(c_1 \cdot \frac{1}{t} + c_2 t^2\right), \tag{29}$$

the real constants c_1 and c_2 will be determined from the initial conditions that we imposed to movement.

2. The parametric oscillations of a mechanical system have as mathematical model the non–linear differential equation

$$\ddot{x} + 2t \cdot \ddot{x} + \left(1 + t^2 + \frac{1}{t^2}\right)x = 0. \tag{30}$$

and for it , we will determine analytical exact solutions.

So

$$x_1 = e^{-\frac{1}{2}\int 2tdt} = e^{-t^2/2};$$
$$I(t) = 1 + t^2 + \frac{1}{t^2} - \frac{1}{4} \cdot 4t^2 - \frac{1}{2} \cdot 2 = \frac{1}{t^2} \tag{31}$$

and the differential equation (30) becomes, according to (19)

$$y'' + \frac{1}{t^2} y = 0, \tag{32}$$

which can be solved as an Euler differential equation, seeking solutions as

$$y = z^r, \quad r \in C. \tag{33}$$

Because the derivations of the first and second order of y are

$$y' = r \cdot z^{r-1};$$
$$y'' = r(r-1) \cdot z^{r-2} \tag{34}$$

after replacing them in equation (28), we obtain the characteristic equation

$$r^2 - r + 1 = 0 \tag{35}$$

with complex concerted roots

$$r_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$
$$r_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}. \tag{36}$$

Because the particular solutions of the differential equation (32) are

$$y_1 = t^{1/2} \cdot \cos\left(\frac{\sqrt{3}}{2}\ln t\right),$$
$$y_2 = t^{1/2} \cdot \sin\left(\frac{\sqrt{3}}{2}\ln t\right), \tag{37}$$

the general solution of the differential equation (32) will be

$$y = t^{1/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right];$$
$$c_1, c_2 \in R \tag{38}$$

and for the differential equation (30) we obtain the analytical exact solution

$$x(t) = t^{1/2} \cdot e^{-t^2/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln t\right) + \right.$$

$$+c_2 \sin\left(\frac{\sqrt{3}}{2}\ln t\right)\Bigg]. \tag{39}$$

The constants c_1, c_2 are calculated like in the previous case.

II. Going back to (12), if

$$x_1'' + p_1(t)\cdot x_1' + p_2(t)\cdot x_1 = 0 \tag{40}$$

then x_1 is a particular solution for the differential equation (1), which can be written

$$x_1 \cdot y'' + \left[2x_1' + p_1(t)x_1\right]\cdot y' = 0 \tag{41}$$

and this permits the reduction of order by one unit after the substitution

$$y' = z,\ z = z(t), \tag{42}$$

then it is solved as an ordinary differential equation with separable variables.

4. Example

The free parametric vibrations of a mechanical system are described by the non-linear homogeneous differential equation

$$\ddot{x} - t^2\dot{x} + \left(t - \frac{2}{t^2}\right)x = 0 \tag{43}$$

and we find exact analytical solutions.

The equation has the particular solution

$$x_1 = \frac{1}{t}, \tag{44}$$

using the equality (41) becomes

$$\frac{1}{t}y'' + \left[\frac{2}{t^2} + t\right]y = 0 \tag{45}$$

and then, after the substitution (42), we write

$$z' - \left(\frac{2}{t} + t^2\right)z = 0, \tag{46}$$

the differential equation with separable variables it is written

$$\frac{z'}{z} = \left(\frac{2}{t} + t^2\right), \tag{47}$$

we integrate both members and the solution will be

$$z = c_1 t^2 e^{\frac{1}{3}t^3},\ c_1 \in R. \tag{48}$$

So according to (42)

$$y' = c_1 t^2 e^{\frac{1}{3}t^3}, \tag{49}$$

after integrations

$$y = c_1 e^{\frac{1}{3}t^3} + c_2,\ c_2 \in R \tag{50}$$

Using the substitution (4) we obtain the analytical exact solution

$$x(t) = \frac{1}{t}\left(c_1 e^{\frac{1}{3}t^3} + c_2\right) \tag{51}$$

the constants c_1 , c_2 being obtained from the initial conditions that were imposed to movement.

5. Conclusion

For the non-linear differential equations of second order that describe the free parametric oscillations of mechanical systems there are situations when we can obtain analytical exact solutions that offer the possibility to study the movement of the respective systems.

References

[1] **Gh. Cautes** - *About the analytical exact solutions of the differential equations of the non-linear oscillators*, The Annals of "Dunarea de Jos" University of Galati, Fascicle XIV Mechanical Engineering, ISSN 1224-5615, 2004

[2] **Gh.Cautes** - *About the free parameter vibrations of the mechanical systems*, The Annals of "Dunarea de Jos" University of Galati, Fascicle XIV Mechanical Engineering, ISSN 1224-5615, 2008

[3]**Gh. Cautes** – *Ecuatii diferentiale. Notiuni teoretice. Aplicatii in mecanica si electricitate.* Editura Academica, ISBN 978-973-8937-08-06, Galati, 2006