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THE PARAMETRICAL FREE VIBRATIONS OF ELASTIC SYSTEMS. THE ANALYTICAL EXACT SOLUTIONS

Assoc.Prof.Math.Dr.Eng. Gh. Cautes
“Dunarea de Jos” University of Galati

ABSTRACT

The movements of many material systems can be described by differential equations that have coefficients that depend on time. It is difficult to determine the solutions of these equations. Although many concrete problems lead to non-linear differential equations where we can also find the term of variable damping, the classic equations that have been studied more were Hill or Mathieu, with periodic coefficient, that do not contain derivations of the first order. At this kind of equations, we reduce them to second order using substitutions as we will demonstrate. In this work, we show that we can obtain analytical exact solutions for non-linear homogeneous differential equations with a variable coefficient which describes the parametric free vibrations of mechanical systems.

KEYWORDS: vibration, mechanical system, non-linear

1. INTRODUCTION

Some mechanical systems have variable parameters in time such as the mass, the dimensions, the elastic coefficient or the damping coefficient. The mathematical pendulum with the variable length is that kind of a system.

Free oscillations of a non-linear mechanical systems with variable parameters are described with homogeneous the equation

$$m\ddot{x} + F_f \cdot \dot{x} + F_{e1} \cdot x = 0 \qquad (1)$$

where m is the mass of the system, F_f the damping force, F_{e1} the elastic force, all of them depending on time, x movement and its derivations in relation to the time. After division to m, the mass appears no more in the differential equation. For the first differential equation, it does’nt exist any method for calculating the exact analytical solutions but for the particular ones.

In [1] for the particular differential equation

$$\ddot{x} + p(t) \cdot x = c \cdot x^{-\alpha} \qquad (2)$$

where $c, \alpha \in R$, there resulted the analytical exact solutions.

In [2] we have shown that we can find analytical approximate solutions for non-linear differential equations as

$$m\ddot{x} - ct^k x = 0;\ c,k \in R \qquad (3)$$

which describe the parametrical free vibrations of the mechanical systems that do not imply damping forces. The theoretical concept that was used to determine solutions employed the theory of Bessel functions.

Next, we will solve a special category of differential equation (1) such as

$$\begin{aligned} & x^m \cdot \dot{x}^n \cdot \ddot{x}^p + F_1(t) \cdot x^n \cdot \dot{x}^p \cdot \ddot{x}^m + \\ & + F_2(t) \cdot x^p \cdot \dot{x}^m \cdot \ddot{x}^n = 0 \end{aligned} \qquad (4)$$

homogeneous for $x, \dot{x}, \ddot{x}$ with $m, n, p \in N$.

We will show, using particular examples, that differential equation (4) represents the general form of some mathematical models which describe the vibrations of real mechanical systems.

2. THEORETICAL CONCEPT

With the substitution,

$$\frac{\dot{x}}{x} = z,\ z = z(t) \qquad (5)$$

and knowing that

$$\begin{aligned} &\dot{x} = x \cdot z, \\ &\ddot{x} = \dot{x}z + x\dot{z} = xz^2 + x\dot{z} = x(z^2 + \dot{z}) \end{aligned} \qquad (6)$$

from (4) it results the differential equation

$$\begin{aligned} &x^m \cdot x^n \cdot z^n \cdot x^p (z^2 + \dot{z})^p + \\ &+ F_1(t) \cdot x^n \cdot x^p \cdot z^p \cdot x^m (z^2 + \dot{z})^m + \\ &+ F_2(t) \cdot x^p \cdot x^m \cdot z^m \cdot x^n (z^2 + \dot{z})^n = 0 \end{aligned} \qquad (7)$$

If $min(m,n,p) = m$ and dividing with $x^{m+n+p} \cdot z^m (z^2 + \dot{z})^m$ we obtain

$$\begin{aligned} &z^{n-m} (z^2 + \dot{z})^{p-m} + F_1(t) \cdot z^{p-m} + \\ &+ F_2(t) \cdot (z^2 + \dot{z})^{n-m} = 0 \end{aligned} \qquad (8)$$

ordinar differential equation of the first order, or Bernoulli differential equation type, or Riccati equation type, or liniar differential equation type depending on the values of m,n,p.

Calculating $z(t)$, from the substitution, it results

$$z(t) = e^{\int x(t)dt} \qquad (9)$$

after just one integration operation.

In the differential equation (4) can appear expressions with other combinations $x, \dot{x}, \ddot{x}$ on the condition that the differential equation remains homogeneous.

Particular case. The differential equation

$$\ddot{x} - \left[\frac{1}{x} \cdot \dot{x} + f(t) \right] \cdot \dot{x} - g(t) \cdot x = 0 \qquad (10)$$

can be written

$$x\ddot{x} - \dot{x}^2 - f(t) \cdot x\dot{x} - g(t) \cdot x^2 = 0. \qquad (11)$$

Replacing (5) with (6), we obtain after calculations

$$\dot{z} - f(t)z = g(t), \qquad (12)$$

a non-homogeneous differential equation with the general exact analytical solution

$$z = e^{\int f(t)dt} \left[c + \int g(t) \cdot e^{-\int f(t)dt} dt \right], c \in R \qquad (13)$$

and using the substitution, we can write the solution x(t) of the differential equation (10).

3. EXAMPLES

We exemplify now for mechanical systems whose parametrical oscillations are characterized by differential equations (4).

1. The differential equation

$$\ddot{x} - \left(\frac{1}{x} \cdot \dot{x} + \frac{2t}{1-t^2} \right) \dot{x} - \frac{1}{1-t^2} x = 0 \qquad (14)$$

describes a mechanical system with the damping force as variable in the time and non-linear dependent from the speed.

This gives later a differential equation

$$\begin{aligned} &(1-t^2)x \cdot \ddot{x} - (1-t^2)\dot{x}^2 - \\ &- 2t \cdot x \cdot \dot{x} - x^2 = 0 \end{aligned} \qquad (15)$$

From (5) and (6) we obtain

$$\begin{aligned} &(1-t^2) \cdot x \cdot x(z^2 + \dot{z}) - \\ &-(1-t^2)x^2 z^2 - 2t \cdot x \cdot xz - x^2 = 0 \end{aligned} \qquad (16)$$

After dividing to x^2 and a series of calculations, it results

$$(1-t^2)\dot{z} - 2tz = 1, \qquad (17)$$

which means

$$z' - \frac{2t}{1-t^2} z = \frac{1}{1-t^2} \qquad (18)$$

linear differential equation of first order, non-homogeneous, with the general exact analytical solution

$$z(t)=e^{\int\frac{2t}{1-t^2}dt}\left[c_1+\int\frac{1}{1-t^2}\cdot e^{-\int\frac{2t}{1-t^2}dt}dt\right]=$$
$$=e^{-ln(1-t^2)}\left[c_1+\int\frac{1}{1-t^2}\cdot e^{ln(1-t^2)}dt\right]=$$
$$=\frac{1}{1-t^2}(c_1+\int dt)=\frac{1}{1-t^2}(c_1+t). \tag{19}$$

Using (9), it results the general solution of the equation (14)

$$x(t)=e^{\int\frac{c_1+t}{1-t^2}dt}=$$
$$=e^{c_1\cdot\frac{1}{2}ln\frac{t+1}{t-1}-\frac{1}{2}ln(1-t^2)+k}=$$
$$=\sqrt{\left(\frac{t+1}{t-1}\right)^{c_1}}\cdot\frac{1}{\sqrt{1-t^2}}\cdot c_2=$$
$$=\sqrt{\frac{(t+1)^{c_1}}{(t+1)^{c_1}\cdot\sqrt{(1-t^2)}}}\cdot c_2 \tag{20}$$

where $c_1, c_2, k\in R$; $c_2=e^k$.

2. The parametrical free vibrations of an elastic mechanical system are described by the non-linear homogenous differential equation

$$\ddot{x}+\frac{1}{x}\cdot\dot{x}^2-\frac{1}{t}\cdot\dot{x}=0 \tag{21}$$

The equation can be written

$$tx\cdot\ddot{x}+t\cdot\dot{x}^2-x\dot{x}=0 \tag{22}$$

and using the replacement (5) it becomes

$$\dot{z}-\frac{1}{t}z=-2z^2 \tag{23}$$

a Bernoulli differential equation.

After dividing with z^2 and writing

$$z^{-1}=u,\ u=u(t) \tag{24}$$

we obtain the linear differential equation

$$\dot{u}+\frac{1}{t}\cdot u=2 \tag{25}$$

with the solution

$$u=\frac{1}{t}(c_1+t^2), \tag{26}$$

and results

$$z(t)=\frac{t}{c_1+t^2},\ c_1\in R\ . \tag{27}$$

The differential equation (21) has a general analytical solution

$$x(t)=e^{\int\frac{tdt}{c_1+t^2}}=e^{\frac{1}{2}ln(c_1+t^2)+k}=$$
$$=c_2\sqrt{c_1+t^2} \tag{28}$$

where $c=e^k$, $k\in R$.

3. The following non-linear differential equation describes parametrical vibrations of a mechanical system

$$\ddot{x}+\frac{1}{x}\cdot\dot{x}^2-\frac{k}{t^2}\cdot x=0,\ k\in R \tag{29}$$

can be written

$$t^2x\ddot{x}+t^2\dot{x}^2-kx^2=0 \tag{30}$$

and it is homogeneous. With substitution (5), we obtain, after simple calculations, the Riccati differential equation

$$\dot{z}=-2z^2+\frac{1}{t^2}\cdot k \tag{31}$$

for which we look for particular solutions such as

$$z=\frac{c}{t},\ c\in R\ . \tag{32}$$

From the verifying condition of the equation, it results

$$2c^2 - c - k = 0 \qquad (33)$$

with the real solutions c_1, c_2 .

The particular solutions of the differential equation (31) will be

$$z_1 = \frac{c_1}{t},\ z_2 = \frac{c_2}{t} \qquad (34)$$

and then from

$$\frac{z - z_1}{z - z_2} = c_3 e^{\int (z_1 - z_2) P(t) dt} \qquad (35)$$

with $c \in R$ and $P(t) = -2$ we can obtain $z(t)$ solution.

From (9), we can calculate solution $x(t)$.

Particular case. For $k = \frac{3}{8}$ from (33) we obtain

$$c_1 = -\frac{1}{4},\ c_2 = \frac{3}{4} \qquad (36)$$

so the particular solutions of the differential equation (31) are

$$z_1(t) = -\frac{1}{4t};\ z_2(t) = \frac{3}{4t}. \qquad (37)$$

The general solution of the equation (31) results [3] from (35) and we obtain

$$z(t) = \frac{3}{4\alpha} \cdot \frac{\alpha \cdot t^2 + 1}{t\left(t^2 - \frac{1}{\alpha}\right)},\ \alpha \in R;\ \alpha > 0 \qquad (38)$$

From (9), after calculating the integral, it results the exact solution of the differential equation (29), for $k = \frac{3}{8}$

$$x(t) = \sqrt[4]{\left|\alpha t - \frac{1}{t}\right|^3} \qquad (39)$$

4. The free parametrical vibrations of an elastic mechanical system are described by the non-linear differential equation

$$\ddot{x} - \left(\frac{2}{x} \cdot \dot{x} - \frac{1}{t}\right) \cdot \dot{x} + \frac{1}{t^2} \cdot x = 0. \qquad (40)$$

With substitution (5), becomes

$$\dot{z} = z^2 - \frac{1}{t} z = -\frac{1}{t^2}, \qquad (41)$$

a Riccati differential equation with the particular solutions

$$z_1 = \frac{1}{t},\ z_2 = -\frac{1}{t}. \qquad (42)$$

The general analytical solution $z(t)$ is obtained from (35) where $c \in R$ and $P(t)$ are the coefficient of z^2 in the Riccati equation.

We obtain

$$z(t) = \frac{c_1 t^2 + 1}{t(1 - c_1 t^2)} \qquad (43)$$

and then

$$x(t) = \frac{c_2 t}{1 - c_1 t^2},\ c_2 \in R \qquad (44)$$

In each example, the real constants were obtained from the initial conditions that were imposed to movement.

5. CONCLUSION

For the non-linear differential equations of second order that describe the free parametrical oscillations of mechanical systems, there are situations when we can obtain analytical exact solutions that offer the possibility to study the movement of the respective systems.

REFERENCES

[1] **Gh. Cautes** - *About the analytical exact solutions of the differential equations of the non-linear oscillators*, The Annals of “Dunarea de Jos” University of Galati, Fascicle XIV Mechanical Engineering, ISSN 1224-5615, 2004.
[2] **Gh.Cautes** - *About the free parameter vibrations of the mechanical systems*, The Annals of “Dunarea de Jos” University of Galati, Fascicle XIV Mechanical Engineering, ISSN 1224-5615,2008.
[3] **Gh. Cautes** – *Ecuatii diferentiale. Notiuni teoretice. Aplicatii in mecanica si electricitate.* Editura Academica, ISBN 978-973-8937-08-06, Galati, 2006.